14. On a Characteristic Property of Completely Normal Spaces

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Let H be a topological space. We shall consider the following condition:

(V) *For any two subsets* X_1 *and* X_2 *of* H, *there exist two closed sets* H_1 *and* H_2 *such that*

(1) $$H=H_1\cup H_2$$

(2) $$H_1\cap H_2\cap(\overline{X}_1\cup\overline{X}_2)=\overline{X}_1\cap\overline{X}_2$$

(3) $$\overline{X}_i\subset H_i, \qquad i=1, 2.$$

Here the bar means the closure operation.

As is well known, the metrizability of H implies (V) and (V) implies the normality of H. However it seems that a closer relation between (V) and the separation axioms has not been given in the literature.[1]

The object of this note is to show that a topological space H is completely normal if and only if it satisfies the condition (V).

1. First we shall prove

Lemma 1. The conditions (1), (2), and (3) imply (1), (3), and (2′) below:

(2′) $$H_i\cap(\overline{X}_1\cup\overline{X}_2)\subset\overline{X}_i, \qquad i=1, 2,$$

and conversely.

Proof. It is obvious that (1), (2′), and (3) imply (2); we have only to prove that (1), (2), and (3) imply (2′). Since $H_1\cap H_2\cap(\overline{X}_1\cup\overline{X}_2)\cap\overline{X}_2=\overline{X}_1\cap\overline{X}_2\cap\overline{X}_2$ by (2), we have $H_1\cap\overline{X}_2=\overline{X}_1\cap\overline{X}_2$ by (3). Therefore $(H_1\cap\overline{X}_2)\cup\overline{X}_1=(\overline{X}_1\cap\overline{X}_2)\cup\overline{X}_1$, and hence $H_1\cap(\overline{X}_1\cup\overline{X}_2)\subset\overline{X}_1$. Similarly we have $H_2\cap(\overline{X}_1\cup\overline{X}_2)\subset\overline{X}_2$.

Next we shall recall the definition of completely normal spaces; a topological space H is said to be completely normal if the following condition is satisfied:

For any two subsets Y_1 and Y_2 of H such that

(4) $$\overline{Y}_1\cap Y_2=Y_1\cap\overline{Y}_2=0,$$

1) Cf. A. D. Wallace: Dimensional types, Bull. Amer. Math. Soc., **51**, 679–681 (1945). Indeed, he said in his paper "the validity of (V) is a well-known property of metric spaces but we have no reference to its formulation in the literature as an axiom".