52. On Homotopy Groups of Function Spaces

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1. Introduction: Since M. Abe defined "Abe groups" [1], various kinds of homotopy groups of function spaces have been introduced by some authors. These groups are considered as homotopy groups of suitable pseudo fibre spaces. Our purpose of this paper is to investigate the homotopy groups of J. R. Jackson [6] and the abhomotopy groups introduced by S. T. Hu [3] from this point of view.

2. Pseudo fibre spaces: By a pseudo fibre space (X, p, B), we understand a collection of two spaces X, B and a continuous mapping $p: X \longrightarrow B$ of X onto B satisfying the "Lifting homotopy theorem" (p. 63, P. J. Hilton [2]; p. 443, J. P. Serre [7]). We assume that X is arcwise connected. The projection $p: X \longrightarrow B$ induces the isomorphism:

(1) $$p'_n: \pi_n(X, X_0) \longrightarrow \pi_n(B), \qquad n \geq 2,$$

and the homomorphism: $p_n: \pi_n(X) \longrightarrow \pi_n(B)$, $n \geq 1$, where X_0 is the fibre over a point $b_0 \in B$. It is well known that the homotopy sequence:

(2) $$\cdots \longrightarrow \pi_{n+1}(B) \xrightarrow{d_{n+1}} \pi_n(X_0) \xrightarrow{i_n} \pi_n(X) \xrightarrow{p_n} \pi_n(B) \longrightarrow \cdots, \qquad n \geq 1,$$

of (X, p, B) is exact. The main results of this paper will be based on the following two theorems.

Theorem 1. *If the pseudo fibre space (X, p, B) admits a cross section, then we have the direct sum relation*

$$\pi_n(X) \approx \pi_n(X_0) + \pi_n(B), \qquad n \geq 2,$$

and $\pi_1(X)$ contains two subgroups M and N such that M is normal and isomorphic to $\pi_1(X_0)$, p_1 maps N isomorphically onto $\pi_1(B)$ and each element of $\pi_1(X)$ is uniquely representable as the product of an element of M with an element of N. (For example, see Theorem 27.6; S. T. Hu [4].)

Theorem 2. *Let (X, p, B) be a pseudo fibre space. If the total space X is deformable into the fibre X_0 relative to a point $x_0 \in X_0$, then we have the direct sum relation (direct product, for $n=1$):*

$$\pi_n(X_0) \approx \pi_{n+1}(B) + \pi_n(X), \qquad n \geq 1.$$

(Proof) For $n \geq 2$, the theorem follows from Theorem 27.10, S. T. Hu [4]. According to the same theorem, $\pi_1(X_0)$ contains two subgroups M and N such that d_2 maps $\pi_2(B)$ isomorphically onto M, i_1 maps N isomorphically onto $\pi_1(X)$ and each element of $\pi_1(X_0)$ is uniquely representable as the product of an element of M and an element of N. Thus the proof is complete, if we prove that N is