51. *On the Property of Lebesgue in Uniform Spaces*

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In this Note, we shall introduce a new concept, the property of Lebesgue in a uniform space.*) Some properties of Lebesgue property in metric spaces were studied in 1950 by A. A. Monteiro and M. M. Peixoto ((2), (3)).

Let S be a topological space. A covering of S is a family of open sets whose union is S. The covering is called *binary* if it consists of two open sets or *finite* if it consists of a finite family of open sets.

Now we shall consider a separated uniform space E. Let $\mathfrak{S}$ be the filter of surroundings of E. If $A\subset E$, and $V\in\mathfrak{S}$, we denote by $V(A)$ the image of the set $(E\times A)\frown V$ by the projection of $E\times E$ onto the first factor E.

We say that a covering $\mathfrak{F}=\{O_\alpha\}$ of E has the *Lebesgue property* if there is a surrounding V of $\mathfrak{S}$ such that, for each x of E, we can find an open set O_α satisfying $V(x)\subset O_\alpha$.

It is clear that, if any finite covering has the Lebesgue property, so is binary covering. We shall prove the following

Theorem 1. If a uniform space induced by $\mathfrak{S}$ is normal and every binary covering has the Lebesgue property, then every finite covering has the Lebesgue property.

Proof. Let $O_i(i=1, 2, \dots, n)$ be a finite covering of E. By the normality of E, we can find a covering $\{G_i\}$ such that $G_i\subset\overline{G}_i\subset O_i$. Therefore $\bigcup_{i=1}^{n} G_i=\bigcup_{i=1}^{n}\overline{G}_i=E$. Let $H_i=E-\overline{G}_i$, then, for each i, $\{O_i, H_i\}$ is a binary covering of E, and it has the Lebesgue property. Let V_i be a surrounding for the covering $\{O_i, H_i\}$, and put $V=\bigcap_{i=1}^{n} V_i$, then $V(x)\subset V_i(x)$ $(i=1, 2, \dots, n)$ for every x of E. Suppose that $V(x)\subset H_i$ $(i=1, 2, \dots, n)$, then

$$V(x)\subset\bigcap_{i=1}^{n} H_i=\bigcap_{i=1}^{n}(E-\overline{G}_i)=E-\bigcup_{i=1}^{n}\overline{G}_i=\text{empty}.$$

Hence there is at least one of i such that $V(x)\subset O_i$. Q.E.D.

To prove that any compact space has the Lebesgue property, we shall show the following

Theorem 2. The following two properties are equivalent:

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*) Throughout this Note, we use the basic concepts of uniform spaces in N. Bourbaki (1). We shall use the terminologies in P. Samuel (4).