48. Some Trigonometrical Series. XII

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1. A. Zygmund [1] has proved the following theorems.

Theorem 1. *Let $a(x)$ be a positive, decreasing and convex function in the interval $(0, \infty)$ such that*

(1) $$a(x) \downarrow 0, \quad xa(x) \uparrow \qquad as\ x \uparrow \infty.$$

Let $a_n = a(n)$ and

(2) $$\bar{f}(x) = \sum_{n=1}^{\infty} a_n \sin nx,$$

then we have

(3) $$\bar{f}(x) \sim x^{-1} a(x^{-1}) \qquad as\ x \to 0.$$

Theorem 2. *Let $a(x)$ be a positive, decreasing and convex function in the interval $(0, \infty)$, tending to zero as $x \to \infty$. Let $a_n = a(n)$ and suppose that*

(4) $$n\Delta a_n \downarrow, \ \sum a_n = \infty.$$

If we put

(5) $$f(x) = \sum_{n=1}^{\infty} a_n \cos nx,$$

then we have

(6) $$f(x) \sim \int_0^{1/x} t\,|a'(t)|\,dt \qquad as\ x \downarrow 0.$$

Omitting the second condition of $a(x)$ in (1), we prove the following

Theorem 3. *Let $a(x)$ be a positive, decreasing and convex function in the interval $(0, \infty)$, tending to zero as $x \to \infty$. Let $a_n = a(n)$ and define $\bar{f}(x)$ by (2), then*

(7) $$\bar{f}(x) \sim x \int_0^{1/x} t a(t) dt \qquad as\ x \to 0,$$

when $\bar{f}(x)$ is not bounded or the right side is ultimately positive.

If the second condition of (1) is satisfied, then we can easily see that (7) becomes (3).

In Theorem 2 we can replace the first condition of (4) by $\Delta^3 a_n \leqq 0$, that is,

Theorem 4. *Let $a(x)$ be a positive, decreasing and convex function in the interval $(0, \infty)$, tending to zero as $x \to \infty$ and let $-a'(t)$ be convex. Let $a_n = a(n)$ and suppose that $\sum a_n = \infty$. Then*

$$f(x) \sim \int_0^{1/x} t\,|a'(t)|\,dt \qquad as\ x \to 0.$$