63. On the Property of Lebesgue in Uniform Spaces. II

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(Comm. by K. KUNUGI, M.J.A., May 13, 1955)

In this Note, we shall discuss the relation between Lebesgue property and uniformly continuity in a uniform space.[*] The theorems to be proved are generalisations of some results by A. A. Monteiro and M. M. Peixoto (3).

Theorem 1. If a uniform space E is normal and every bounded continuous function is uniformly continuous, then any finite covering of E has the Lebesgue property.

Proof. Let F_1, F_2 be two closed sets such that $F_1 \frown F_2 = 0$. By a theorem of Urysohn, we can find a continuous function $f(x)$ on the uniform space E such that

(1) $0 \leq f(x) \leq 1$ on E,

(2) $f(x)=0$ for $x \in F_1$,

and

(3) $f(x)=1$ for $x \in F_2$.

Since the function $f(x)$ is uniform continuous, for a given positive number ε less than 1, there is a surrounding V such that $V(a) \ni x, y$ implies

(4) $|f(x)-f(y)| < \varepsilon$.

Suppose that $V(F_1) \frown F_2 \neq 0$, then, for $x \in V(F_1) \frown F_2$, $y \in F_2$, $(x, y) \in V$, and $x \in F_2$, and hence $|f(x)-f(y)| < \varepsilon$ by (4). From (2) and (3) $|f(x)-f(y)|=1$, which is a contradiction. Therefore any binary covering of E has the property of Lebesgue, and since E is normal, each finite covering of E has the Lebesgue property. Q.E.D.

Conversely, we shall prove the following

Theorem 2. If any covering of a uniform space E has the Lebesgue property, then any continuous function on E is uniformly continuous.

Proof. Let $f(x)$ be a continuous function on E. To prove that $f(x)$ is uniformly continuous, let $O_\alpha = f^{-1}(I_\alpha)$, where I_α is any open interval with the length ε. $\{O_\alpha\}$ is an open covering of E. Since E has the Lebesgue property, there is a surrounding V such that $V(a) \subset O_\alpha$ for some index α depending on a. Hence $V(a) \ni x, y$ implies

$$|f(x)-f(y)| \leq |f(x)-f(a)| + |f(a)-f(y)| < 2\varepsilon.$$

This shows that $f(x)$ is uniformly continuous.

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*) For the definitions and properties of Lebesgue property in a uniform space, see K. Iséki (2). For the definition of uniformly continuity, see N. Bourbaki (1) or G. Nöbeling (4).