58. On the π-Regularity of Certain Rings

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In his paper [2][1] G. Azumaya introduced the notions of right, left, and strong π-regularities of rings (and of elements in a ring), and investigated connections between such types of rings, some of which had previously been studied by Kaplansky and others.[2] Recently one of the present authors obtained several properties on such rings under the assumption that the given ring is of bounded index (see [5]).

In the present note we shall generalize some results obtained in the papers remarked above by showing that they are applicable to some wider class of rings which contains, for example, rings with polynomial identities in the sense of Levitzki [3].

For the sake of convenience we insert here some definitions which are fundamental in our considerations:

An element a of a ring R is said to be *π-regular* in R if there exist an element x in R and a positive integer n such that $a^nxa^n=a^n$, and if there exist an x and an n such that $a^{n+1}x=a^n(xa^{n+1}=a^n)$ then a is said to be *right* (*left*) *π-regular*. An element which is right as well as left π-regular is said to be *strongly π-regular*. We say that R is a π-regular ring if every element of R is π-regular. Right, left, and, strongly π-regular rings are defined similarly. That a ring is *of bounded index* means that the least upper bound of all indices of nilpotent elements in the ring (=index of the ring) is finite.

1. Nil-ideals of bounded index. We consider first the following ring-property:

($*$) A ring is nil and of bounded index.

Theorem 1. *The ring-property* ($*$) *is an additive* $F\mathfrak{S}$*-property:*[3]
(*E1*) *Each right* (*left*) *ideal in a* ($*$)*-ring is a* ($*$)*-right* (*left*) *ideal.*
(*E2*) *If* A *is a* ($*$)*-right* (*left*) *ideal in a ring* R, *then* rA (Ar) *is a* ($*$)*-right* (*left*) *ideal, where* $r\in R$.
(*E3*) *If* $R^2=0$, *then* R *is a* ($*$)*-ring.*
(*E4*) *If* A *is an ideal*[4] *of* R *such that* $A^2=0$, *then* R *is a* ($*$)*-ring if and only if* R/A *is so.*

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1) Numbers in brackets refer to the references at the end of this paper.
2) See, for example, the bibliography cited in [2].
3) See [4, § 2].
4) The term "ideal" will mean a two-sided ideal throughout this paper.