

105. On the Property of Lebesgue in Uniform Spaces. III

By Kiyoshi ISÉKI

Kobe University

(Comm. by K. KUNUGI, M.J.A., July 12, 1955)

In my previous Note (3),¹⁾ we proved a theorem: *If a uniform space E is normal and every bounded continuous function is uniformly continuous, then any finite open covering of E has the Lebesgue property.*

First of all, following his valuable advice of Prof. Junji Hashimoto, we shall state the following

Theorem 1. If a uniform space E is normal, and any finite open covering has the Lebesgue property, then every bounded continuous function is uniformly continuous.

The proof is very similar to Theorem 2 of my Note (4), and is contained in it. Therefore it will be omitted.

Throughout the remainder of this Note, we shall give a generalisation of a theorem by A. A. Monteiro and M. M. Peixoto (5), and, as its application, we shall give conditions that uniform spaces be compact (in the sense of N. Bourbaki).

Theorem 2. Let E be a separated uniform space. If any open covering of E has the Lebesgue property and E is precompact,²⁾ then E is compact.

The converse of Theorem 2 is clear from Theorem 3 in my Note (3).

Proof. Let $\mathfrak{F} = \{O_\alpha\}$ be an open covering of E . Since the covering \mathfrak{F} has the Lebesgue property, there is a surrounding V such that $V(x) \subset O_\alpha$, where α depends on x . From the precompactness of E , we can find a finite collection A_i ($i=1, 2, \dots, n$) of E such that $A_i \times A_i \subset V$ ($i=1, 2, \dots, n$) and $\bigcup_{i=1}^n A_i = E$. If $x \in A_i$, then $A_i \subset V(x)$, and hence for each i , there is an index α_i such that $A_i \subset V(x) \subset O_{\alpha_i}$. Therefore, since A_i ($i=1, 2, \dots, n$) is a covering of E , $\bigcup_{i=1}^n O_{\alpha_i} = E$, which prove Theorem 2.

From Theorem 2, we shall have the following

Theorem 3. A necessary and sufficient condition for a separated uniform space E to be compact is that every open covering of E has Lebesgue property and every continuous function of E reaches upper bound.

1) For the undefined terminologies, see my two Notes (3), (4).

2) See the definition in N. Bourbaki (1), Chapter 2.