101. On the Group of Conformal Transformations of a Riemannian Manifold

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Let M be a connected and differentiable Riemannian manifold of dimension $n(\geqq 3)$ with the fundamental metric tensor field G. A differentiable homeomorphism φ of M onto M is called a *conformal transformation* if $(\varphi G)_p=\rho(p)G_p$ at every point p of M where ρ is a positive function on M determined by φ and called the *associated function* of φ. If in particular the function ρ is constant, φ is called a *homothetic transformation*. And if furthermore $\rho=1$ at every point of M, φ is said to be an *isometric transformation* or *isometry*.

We denote by $K(M)$, $H(M)$, and $I(M)$ the group of all conformal transformations, that of all homothetic ones and that of all isometries of M respectively. It is then clear that we have $K(M)\supset H(M)\supset I(M)$. As is well known a conformal transformation leaves invariant the Weyl's conformal curvature tensor field C of M.

Now we denote by $K_p(M)$ the group of isotropy of $K(M)$ at a point p of M. If $\varphi\in K_p(M)$, φ induces a linear transformation $\tilde{\varphi}$ on the tangent vector space T_p at p. This correspondence $\varphi\rightarrow\tilde{\varphi}$ is a linear representation[1] of $K_p(M)$ onto $\tilde{K}_p(M)$ which is a subgroup of the homothetic group $H(n)$ of T_p. If in particular $\tilde{K}_p(M)$ is contained in the orthogonal group $O(n)$ of T_p, p is called to be an *isometric point*. If $\tilde{K}_p(M)$ is not contained in $O(n)$, p is said a *homothetic point*.

We shall first establish

THEOREM 1. *The conformal curvature tensor field C of M vanishes at any homothetic point.*

This theorem will be obtained as a corollary to the following lemma.

LEMMA. *Let V be an n-dimensional vector space over the real number field and $\tilde{\varphi}$ a homothetic transformation which is not an orthogonal one. If a tensor S of type (p, q), $p\neq q$, is invariant by $\tilde{\varphi}$, then S is the zero tensor.*

PROOF. We regard S as a multilinear mapping of

$$\underbrace{V\times\cdots\times V}_{p}\times\underbrace{V^*\times\cdots\times V^*}_{q}$$

1) In general this linear representation is not faithful.