# 100. *On a Homogeneous Space with Invariant Affine Connection*

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1. **Preliminaries.** Let $G/H$ be a reductive homogeneous space,[1] where $G$ is a connected Lie group and $H$ a closed subgroup of $G$. Then in the Lie algebra $\mathfrak{g}$ of $G$ there exists a subspace $\mathfrak{m}$ such that $\mathfrak{g}=\mathfrak{m}+\mathfrak{h}$ (direct) and $\mathrm{ad}(H)\cdot\mathfrak{m}\subset\mathfrak{m}$, where $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}$ corresponding to $H$. If we denote by $[\mathfrak{h}, \mathfrak{m}]$ the subspace spanned by all elements of the form $[U, X]$, $U\in\mathfrak{h}$, $X\in\mathfrak{m}$, we have then $[\mathfrak{h}, \mathfrak{m}]\subset\mathfrak{m}$. $\mathfrak{m}$ may be identified with the tangent space at the point $p_0=H$ of $G/H$. Throughout this note we assume that $G$ is almost effective on $G/H$ as a transformation group, that is to say that $H$ does not contain any positive-dimensional normal subgroup of $G$. It follows then that $\mathfrak{h}$ contains no non-trivial ideal of $\mathfrak{g}$ and that the adjoint representation $\mathfrak{h}\rightarrow\mathrm{ad}(\mathfrak{h})$ in $\mathfrak{m}$ of $\mathfrak{h}$ is faithful.

Now let $H$ be a Lie group and $\rho$ a representation of $H$ on a vector space $\mathfrak{m}$. A vector $X$ of $\mathfrak{m}$ is said to be *invariant by $H$* if we have $\rho(h)X=X$ for every $h\in H$. This being the case, we have $\tilde{\rho}(U)\cdot X=0$ for every $U\in\mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of $H$ and $\rho$ is the representation of $\mathfrak{h}$ induced by $\rho$. In this case, $X$ is called *invariant by $\mathfrak{h}$*.

LEMMA. *Let $\mathfrak{h}$ be a Lie algebra and $\rho$ a representation of $\mathfrak{h}$ on a vector space $\mathfrak{m}$. Assume further that $\rho$ is semi-simple. Then there is no non-zero vector of $\mathfrak{m}$ invariant by $\mathfrak{h}$ if and only if $\rho(\mathfrak{h})\cdot\mathfrak{m}=\mathfrak{m}$, where $\rho(\mathfrak{h})\cdot\mathfrak{m}$ is the space spanned by all elements of the form $\rho(U)\cdot X$, $U\in\mathfrak{h}$, $X\in\mathfrak{m}$.*

PROOF. Assuming that there is an invariant vector $X_1$ of $\mathfrak{m}$, let $\mathfrak{m}_1$ be the subspace spanned by $X_1$. Then we have $\rho(\mathfrak{h})\cdot\mathfrak{m}_1=0$. $\rho$ being semi-simple, there exists an invariant subspace $\mathfrak{m}_2$ such that $\mathfrak{m}$ is the direct sum of $\mathfrak{m}_1$ and $\mathfrak{m}_2$, $\rho(\mathfrak{h})\cdot\mathfrak{m}_2\subset\mathfrak{m}_2$. Thus we have

$$\rho(\mathfrak{h})\cdot\mathfrak{m}=\rho(\mathfrak{h})\cdot(\mathfrak{m}_1+\mathfrak{m}_2)=\rho(\mathfrak{h})\cdot\mathfrak{m}_2\subset\mathfrak{m}_2\neq\mathfrak{m}.$$

Conversely, assume that $\rho(\mathfrak{h})\cdot\mathfrak{m}=\mathfrak{m}_2\neq\mathfrak{m}$, then the subspace $\mathfrak{m}_2$ is a proper invariant subspace of $\mathfrak{m}$ and there exists an invariant subspace $\mathfrak{m}_1$ such that $\mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2$ (direct). Then we have $\rho(\mathfrak{h})\cdot\mathfrak{m}_1\subset\mathfrak{m}_1\cap\mathfrak{m}_2=(0)$, which proves that $\mathfrak{m}$ has an invariant vector.

2. **The property** (A). The notation and assumptions being as

1) Cf. K. Nomizu: Invariant affine connections on homogeneous spaces, Amer. J. Math., **76**, 33–65 (1954).