96. A Characterization of the Second Order Elliptic Differential Operators

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§ 1. **Introduction and the theorem.** We consider the elliptic differential operator

$$(Ef)(x)=a^{ij}(x)\frac{\partial^2 f}{\partial x^i\partial x^j}+b^i(x)\frac{\partial f}{\partial x^i}\ (a^{ij}(x)\xi_i\xi_j\geqq 0) \tag{1.1}$$

as a linear operator from real-valued continuous functions $f(x)$ of the point $x=(x^1,\dots,x^m)$ of an m-dimensional C^∞ manifold R to real-valued continuous functions $(Ef)(x)$ of the point x. As was stressed by W. Feller,[1] it enjoys two important properties:

(1.2) *The local property* which means that the value $(Ef)(x_0)$ is determined by the values of $f(x)$ in any small neighbourhood of x_0.

(1.3) *The property* (E) which says that if $f(x)$ has a local minimum at x_0 then $(Ef)(x_0)\geqq 0$.

W. Feller[2] has determined, for the case of one-dimensional R, the most general class of operators with these two characteristic properties. According to him such operator E can, under the condition $E\cdot 1=0$, be represented by means of repeated differentiation

$$D_vD_uf \tag{1.4}$$

with respect to monotone non-decreasing functions u and v (of these two functions, u is a continuous function).

It is desirable to extend Feller's result to the case of higher dimensional R. The purpose of the present note is to give, by refining the method in a preceding note,[3] a partial answer to this problem. Our result gives a characterization of, so to speak, "the smallest closed extension" A of the differential operator E.

Let us formulate the situation precisely. We assume that (i) *R is a homogeneous Riemann space*, viz. R is a C^∞ Riemann space

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1) W. Feller: The general diffusion operator and positivity preserving semi-groups in one dimension, Ann. Math., **60**, No. 3, 417–436 (1954). W. Feller: On second order differential operators, Ann. Math., **61**, No. 1, 90–105 (1955).

2) See the reference referred to in 1).

3) K. Yosida: On Brownian motion in a homogeneous Riemannian space, Pacific Journ. Math., **2**, No. 2, 263–270 (1952). In this paper, the function space $C(R)$ can be corrected to be the Banach space which is the closure, by the norm defined by the maximum of the absolute value of the function, of the totality of continuous functions on R with compact supports.