95. Lacunary Fourier Series. I

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1. M. E. Noble [1] has proved the following

Theorem N. *If the Fourier series of $f(t)$ has a gap $0<|n-n_k| \leqq N_k$ such that*

$$\lim N_k/\log n_k = \infty$$

and $f(t)$ satisfies a Lipschitz condition of order $\alpha\,(0<\alpha<1)$ in some interval $|t-t_0|\leqq\delta$, then

$$a_{n_k}=O(1/n_k^\alpha), \quad b_{n_k}=O(1/n_k^\alpha),$$

where a_{n_k}, b_{n_k} are non-vanishing Fourier coefficients of $f(t)$.

In the present paper we treat the Fourier series with a certain gap and satisfying some continuity condition at a point, instead of in a small interval. Our theorems depend on the lemma (Lemma 1 in §2), which is due to M. E. Noble, except (iv) and (v).

We can also prove theorems concerning absolute convergence of Fourier series with the above-mentioned conditions, analogously to M. E. Noble [1]. These will be found in the second paper.

2. **Lemma 1.** *Let (δ_m) be a sequence tending to zero and let $n=[4em/\delta_m]$. Then there exists a trigonometrical polynomial $T_n(x)$ of degree not exceeding n with constant term 1 such that:*[1]

(i) $|T_n(x)|\leqq A/\delta_m$, *for all x,*

(ii) $|T_n(x)|\leqq An/\delta_m e^m$, $(\delta_m\leqq|x|\leqq\pi)$,

(iii) $|T_n'(x)|\leqq An/\delta_m$, *for all x,*

(iv) $|T_n'(x)|\leqq A(n^2/\delta_m e^m+1/x^2)$, $(\lambda\delta_m\leqq|x|\leqq\pi,\ \lambda>1)$[2]

(v) $|T_n''(x)|\leqq An^2/\delta_m$, *for all x.*

Proof. Let $E_m=(-\delta_m,\delta_m)$, and $C_m(x)$ be its characteristic function. We choose then $\tau_m=\delta_m/2m$ and construct a set of even function $h_i(x)$ $(i=0,1,2,\dots)$ defined by

$$h_0(x)=\frac{\pi}{\delta_m}C_m(x),$$

$$h_{i+1}(x)=\frac{1}{\tau_m}\int_x^{x+\tau_m}h_i(t)\,dt \qquad (i=0,1,2,\dots),$$

for $x\geqq0$ and $i\leqq m-1$.

It is easy to see that

$$h_m(x)=\begin{cases}0 & (\delta_m\leqq|x|\leqq\pi),\\ \pi/\delta_m & (|x|\leqq\delta_m/2),\end{cases}$$

1) A denotes an absolute constant which is not the same in different occurrences.

2) λ may be taken as near 1 as we like when m is sufficiently large.