145. Note on Commutator Subgroups of Factorisable Groups

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1. A group G is called factorisable by its subgroups A and B if each element g of G may be written in the form $g=ab$ with a in A, b in B (written $G=AB$). Since G. Zappa [1] considered factorisable groups, many interesting results have been obtained by several authors. It seems to the author, however, that the structure of commutator subgroups of factorisable groups is not well known. In this note we shall study the structure of derived groups of factorisable groups by certain subgroups. Let (A, B) denote the subgroup generated by all the elements of the form $aba^{-1}b^{-1}$, where a and b run all the elements of A and B respectively. Our main result which plays an important rôle in the sequel study is stated as follows: *if a group G is factorisable by two subgroups A and B, then the subgroup (A, B) is always normal in G.*

2. **Theorem 1.** *Let G be a factorisable group by two arbitrary subgroups A and B, i.e. $G=AB$. Then the subgroup (A, B) is normal in G.*

Proof. Let α be an element of A and let $aba^{-1}b^{-1}$ be an element of $(A, B)=(B, A)$, where $a\in A$ and $b\in B$. Then we have

$$\alpha^{-1}aba^{-1}b^{-1}\alpha=\alpha^{-1}aba^{-1}\alpha b^{-1}b\alpha^{-1}b^{-1}\alpha$$
$$=(\alpha^{-1}a)b(\alpha^{-1}a)^{-1}b^{-1}\cdot b\alpha^{-1}b^{-1}\alpha.$$

Therefore, this element belongs to (A, B). Similarly, for any element β of B, we obtain that

$$\beta^{-1}aba^{-1}b^{-1}\beta=\beta^{-1}a\beta a^{-1}\cdot a(\beta^{-1}b)a^{-1}(\beta^{-1}b)^{-1};$$

hence $\beta^{-1}aba^{-1}b^{-1}\beta\in(A, B)$, q.e.d.

Lemma. *Let G be a factorisable group by an abelian subgroup A and an arbitrary subgroup B, and let G' and B' be the commutator subgroups of G and B respectively. Then*

$$G'=B'(A, B).$$

Proof. It is obvious that $G'=(AB, AB)\supseteq B'(A, B)$. We shall prove $G'\subseteq B'(A, B)$. Let g and h be two elements of G such that $g=ab$, $h=a'b'$, $a, a'\in A$, $b, b'\in B$. Since there exist α and β such that $ba'=\alpha\beta$, $\alpha\in A$, $\beta\in B$, we have

$$\begin{aligned}(g, h)&=ghg^{-1}h^{-1}\\ &=aba'b'b^{-1}a^{-1}b'^{-1}a'^{-1}\\ &=(a\alpha)(\beta b'b^{-1})a^{-1}b'^{-1}a'^{-1}\\ &\equiv(\beta b'b^{-1})(a\alpha)a^{-1}b'^{-1}a'^{-1} \qquad (\text{mod. } (A, B))\\ &=\beta b'b^{-1}\alpha b'^{-1}a'^{-1}\end{aligned}$$