164. On Countably Paracompact Spaces

By Fumie ISHIKAWA
Osaka Woman's University, Osaka

(Comm. by K. KUNUGI, M.J.A., Dec. 12, 1955)

This note will give a characterization of a countably paracompact space, i.e. a topological space X which has a locally finite refinement for every countable open covering.

Theorem: In order that a topological space X be countably paracompact it is necessary and sufficient that:

A. If a decreasing sequence $\{F_i\}$ of non empty closed sets F_i with vacuous intersection is given, then there exists a decreasing sequence $\{G_i\}$ of open sets such that their closure $\overline{G}_i$ have a vacuous intersection and $G_i \supset F_i$.

Proof. If X is a countably paracompact space and given $\{F_i\}$, then
$\{X-F_i\}$ is a countable open covering of X, therefore it has a locally finite refinement $\mathfrak{W}$. For each open set W of $\mathfrak{W}$ let $g(W)$ be the first $X-F_i$ containing W, and let V_i be the union of all W for which $g(W)=X-F_i$. Then V_i is open and $V_i \subset X-F_i$, $\{V_i\}$ is a locally finite covering of X.

Put: $G_i=\sum_{n=i+1}^{\infty} V_n$. Then G_i is open,

$$G_i \supset X-(V_1+\cdots+V_i) \supset X-(X-F_i)=F_i,$$

hence $G_i \supset F_i$.
For every point x of X there exists a neighborhood $u(x)$ such that it meets only a finite number of V_i, since $\{V_i\}$ is a locally finite. Therefore there exists an i such that $u(x) \frown \sum_{n=i+1}^{\infty} V_n=0$, that is, $x \,\overline{\in}\, \overline{G}_i$. Accordingly $\pi\overline{G}_i=0$.

Conversely let X be a space with the condition A, and let $\{U_i\}$ be a given countable open covering of X.

Put: $F_i=X-\sum_{n=1}^{i} U_n$. Then by the condition A there exist open sets G_i such that $G_i \supset F_i$,

$$G_1 \supset G_2 \ldots \quad \text{and} \quad \pi\overline{G}_i=0.$$

Put: $X-G_i=E_i$. Then E_i is obviously closed and $E_i \subset \sum_{n=1}^{i} U_n$.

Finally put: $V_i=U_i-E_{i-1}$. Then V_i is clearly open and $V_i \subset U_i$. Moreover, since $V_i=U_i-E_{i-1} \supset U_i-\sum_{n=1}^{i-1} U_n$, we have $\sum_{i=1}^{\infty} V_i \supset \sum_{i=1}^{\infty}(U_i-\sum_{n=1}^{i-1} U_n)=\sum_{i=1}^{\infty} U_i=X$, thus $\{V_i\}$ is a refinement of $\{U_i\}$. To each point x of X, we choose the first i such that $x \,\overline{\in}\, \overline{G}_i$. Then there exists a