163. On the Reflexivity of Semi-Continuous Norms

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Let R be a semi-regular space and $\overline{R}$ its conjugate space. In [1] and [2], H. Nakano proved that every continuous pseudo-norm $\|x\|$ is *reflexive*, i.e. $\|a\| = \sup_{\|\overline{x}\| \leqq 1,\, \overline{x} \in \overline{R}^{\|}} |\overline{x}(a)|$ for every $a \in R$ (*cf.* [1] Theorem 32.9 and [2] Theorem 8.4). In that assertion, the hypothesis of the continuity can be weakened, and it will be also discussed with some topological argument in another paper (I. Amemiya and T. Mori: Topological structures in ordered linear spaces, unpublished). Now in this paper we shall give a proof based upon some results in [1].

Theorem. Every semi-continuous pseudo-norm $\|x\|$ *is reflexive.*

Proof. By virtue of the semi-continuity of $\|x\|$, we can suppose that R is regular and $\overline{R} \ni \overline{a} \geqq 0$ is complete in R.

Putting

$$\|a\|_\nu = \inf_{x, y \geqq 0,\, x+y=|a|} \operatorname{Max}\{\|x\|, \nu\overline{a}(y)\} \qquad (\nu = 1, 2, \ldots),$$

we have evidently

$$\nu\overline{a}(|a|) \geqq \|a\|_\nu \quad (\nu = 1, 2, \ldots), \qquad \|a\| \geqq \|a\|_\nu \uparrow_{\nu=1}^{\infty},$$

and hence every $\|x\|_\nu$ is continuous and reflexive.

For every $\alpha > \|a\|_\nu$ $(\nu = 1, 2, \ldots)$ there exist $x_\nu, y_\nu \in R$ $(\nu = 1, 2, \ldots)$ such that

$$x_\nu, y_\nu \geqq 0, \quad x_\nu + y_\nu = |a|, \quad \|x_\nu\| < \alpha, \quad \overline{a}(y_\nu) < \frac{\alpha}{\nu} \quad (\nu = 1, 2, \ldots).$$

By [1] Theorem 33.1, we have

$$s\text{-ind-}\lim_{\nu\to\infty} y_\nu = 0,$$

and hence we obtain by [1] Theorem 33.2

$$\|a\| \leqq \varliminf_{\nu\to\infty} \|x_\nu\| \leqq \alpha.$$

Consequently we have

$$\|a\| = \sup_{\nu=1,2,\ldots} \|a\|_\nu.$$

From the reflexivity of every $\|x\|_\nu$, we can conclude

$$\|a\| \geqq \sup_{\|\overline{x}\| \leqq 1,\, \overline{x} \in \overline{R}^{\|}} |\overline{x}(a)| \geqq \sup_{\nu=1,2,\ldots} \sup_{\|\overline{x}\| \leqq 1,\, \overline{x} \in \overline{R}^{\|\nu}} |\overline{x}(a)| \geqq \sup_{\nu=1,2,\ldots} \sup_{\|\overline{x}\|_\nu \leqq 1,\, \overline{x} \in \overline{R}^{\|\nu}} |\overline{x}(a)| = \|a\|.$$

By this theorem, we obtain immediately (*cf.* [2])

Corollary. The strong topology of R *is reflexive and coincides with the topology of the uniform convergence on the weakly bounded sets in* $\overline{R}$.