

162. On the Commutativity of Projection Operators

By Takasi ITÔ

Hokkaidô University

(Comm. by K. KUNUGI, M.J.A., Dec. 12, 1955)

Let P and Q be projection operators of closed linear manifolds M and N in a Hilbert space R , and $P \cup Q$ be a projection operator of the least closed linear manifold containing both M and N . It is known¹⁾ that if P and Q are commutative: $PQ=QP$, then we have for every $x \in R$

$$\|P \cup Qx\|^2 \leq \|Px\|^2 + \|Qx\|^2.$$

In this paper we shall show that this necessary condition for $PQ=QP$ is sufficient, too.

Theorem 1. For two projection operators P and Q , the following conditions are equivalent.

1) $PQ=QP$.

2) $\|P \cup Qx\|^p \leq \|Px\|^p + \|Qx\|^p$ ($x \in R$) for all p in $0 < p \leq 2$.

3) $\|P \cup Qx\|^p \leq \|Px\|^p + \|Qx\|^p$ ($x \in R$) for some p in $0 < p \leq 2$.

Proof. If $PQ=QP$, then we have $\|P \cup Qx\|^2 \leq \|Px\|^2 + \|Qx\|^2$ ($x \in R$). In the proof of 2) without loss of generality we can suppose $P \cup Q = I$ and $\|x\|=1$. Therefore, we have $\|P \cup Qx\|^p = 1 = \|P \cup Qx\|^2 \leq \|Px\|^2 + \|Qx\|^2 \leq \|Px\|^p + \|Qx\|^p$ for all p in $0 < p \leq 2$. Next we prove that 3) implies 1). As $(P \cup Q)Px = Px$, $(P \cup Q)QPx = QPx$, and $Px = QPx + (I - Q)Px$, we have $(P \cup Q)(I - Q)Px = (I - Q)Px$. Therefore, we conclude $\|(I - Q)Px\|^p = \|P \cup Q(I - Q)Px\|^p \leq \|P(I - Q)Px\|^p + \|Q(I - Q)Px\|^p = \|P(I - Q)Px\|^p$, and hence $\|(I - Q)Px\| \leq \|P(I - Q)Px\|$, so that $(I - Q)P = P(I - Q)P$, that is, $QP = PQP$. Therefore, we obtain $PQ = (QP)^* = (PQP)^* = PQP = QP$.

Theorem 2. For two projection operators P and Q , the following conditions are equivalent.

1) $P \leq Q$ or $Q \leq P$.

2) $\|P \cup Qx\| = \max\{\|Px\|, \|Qx\|\}$ ($x \in R$).

3) $\|P \cup Qx\|^p \leq \|Px\|^p + \|Qx\|^p$ ($x \in R$) for all $p > 2$.

4) $\|P \cup Qx\|^p \leq \|Px\|^p + \|Qx\|^p$ ($x \in R$) for some $p > 2$.

Proof. It is evident that 1) implies 2), 2) implies 3), and 3) implies 4), because $\max\{\|Px\|, \|Qx\|\} \leq (\|Px\|^p + \|Qx\|^p)^{1/p}$. We prove that 4) implies 1). By the similar method as in Theorem 1 we obtain $PQ=QP$ from 4). Therefore, if we have not 1) for P and Q , there are x_1, x_2 , and x_3 such that $P(I - Q)x_1 = x_1 \neq 0$, $Q(1 - P)x_2$

1) H. Nakano: Spectral theory in the Hilbert space, Tokyo Math. Book Series, 4 (1953), Theorem 12.7.