7. Notes on Topological Spaces. I. A Theorem on Uniform Spaces

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The object of this Note is to give a formulation of a theorem on metric space. We suppose that *all spaces considered are separated.*

Let X be a uniform space. The space X is called an *absolute closed* if whenever X is topologically imbedded in a uniform space Y, then X is a closed in Y.

We can easily obtain the following

Theorem 1. A uniform space is absolutely closed if and only if it is complete in sense of uniform structure.

Proof. Let X be an absolutely closed uniform space, and $\hat{X}$ the completion of X. Then $X\subset\hat{X}$ and X is closed in $\hat{X}$. Therefore, by a well-known proposition (N. Bourbaki [1], p. 149, Prop. 6, or G. Nöbeling [2], p. 200, 26.4), X is complete.

Conversely, let X be a complete uniform space, and suppose that X is imbedded in a uniform space Y. Since the same proposition 6 of N. Bourbaki stated in the first part of proof, and Y is separated, X is closed. Therefore X is absolutely closed.

From Theorem 1, we can reduce the following interesting

Theorem 2. Let X be a uniform space and Z any uniform space containing X. If there is a complete uniform space Y containing X and X is a G_δ-set in Y, then X is the intersection of a closed set and a G_δ-set of Z.

Proof. Let Y be a complete uniform space satisfying the condition of Theorem 2. Let Z be any uniform space containing X. Since X is a G_δ-set in Y, there are countable closed sets F_n of Y such that $X=Y-\bigcup_{n=1}^{\infty}F_n$. By Theorem 1, Y is absolutely closed. On the other hand, if we let $Y\cup Z$ be a uniform space, Y is closed in $Y\cup Z$. Hence each closed set F_n is closed in $Y\cup Z$ and therefore F_n and Y are closed in Z.

The identity

$$X=Y\frown\bigcap_{n=1}^{\infty}(Z-F_n)$$

implies that X is the intersection of a closed set and a G_δ-set of Z.

Q.E.D.

Conversely, we have easily seen the following