5. Some Trigonometrical Series. XVIII

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1. Let

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx), \tag{1}$$

$$g(x)\sim\frac{a_0'}{2}+\sum_{n=1}^{\infty}(a_n'\cos nx+b_n'\sin nx). \tag{2}$$

The Parseval relation of $f(x)$ and $g(x)$ is

$$\frac{1}{\pi}\int_0^{2\pi}f(x)g(x)dx=\frac{a_0a_0'}{4}+\sum_{n=1}^{\infty}(a_na_n'+b_nb_n'). \tag{3}$$

The known conditions of $f(x)$ and $g(x)$ for the validity of (3) are as follows [1]:

(i) $f\,\varepsilon\, L^p,\ g\,\varepsilon\, L^{p'},\ p>1,\ 1/p+1/p'=1.$

(ii) $f\,\varepsilon\, L,\ g\,\varepsilon\, B.$

(iii) $f\,\varepsilon\, C,\ g\,\varepsilon\, S.$

In the case (i), the right side of (3) converges but in the cases (ii) and (iii), it is summable $(C,1)$. It is known that, if

(ii′) $f\,\varepsilon\, L,\ g\,\varepsilon\, BV$, or (ii″) $f\,\varepsilon\, Z,\ g\,\varepsilon\, B$,

then the Parseval relation (3) holds and the right side is convergent.

We shall here prove that, in the cases (ii) and (iii) the right side of (3) converges under some additional conditions of $f(x)$ or $g(x)$.

On the other hand, let

$$f(x)\sim\sum_{n=-\infty}^{\infty}c_ne^{inx},\quad g(x)\sim\sum_{n=-\infty}^{\infty}c_n'e^{inx}$$

and $f_\alpha(x)$ be the αth integral of $f(x)$. G. H. Hardy and J. E. Littlewood [2] proved that if

(iv) $f\,\varepsilon\, L^p,\ g\,\varepsilon\, L^q,\ p<2,\ q\leqq p'\quad(1/p+1/p'=1)$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}f_\alpha(x)g(-x)dx=\sum_{n=-\infty}^{\infty}\frac{e^{-\frac{1}{2}\alpha\pi i\,\mathrm{sgn}\,n}}{|n|^\alpha}c_nc_n'\quad\text{for }\alpha=\frac{1}{p}+\frac{1}{q}-1. \tag{4}$$

The case (iv) corresponds to the case (i) of (3). We shall prove (4) in the case corresponding to (ii).

For the proof of our theorem we use the following theorem due to one of us [3] (cf. [4]).

Theorem A. *If*

$$\frac{1}{\delta}\int_0^{\delta}|f(x+t)-f(x)|\,dt=o(1)\qquad(\delta\to 0) \tag{5}$$

for an x and