3. *Closed Mappings and Metric Spaces*

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A mapping of a topological space X onto another topological space Y is said to be closed if the image of every closed subset of X is closed in Y. Concerning the problem: "Under what condition is the image of a metric space under a closed continuous mapping metrizable?", several interesting results have been obtained recently by G. T. Whyburn [6], A. V. Martin [3], and V. K. Balachandran [1]. In the present note, we shall give an answer to this problem by proving that the image space Y of a metric space X under a closed continuous mapping f is metrizable if and only if the boundary $\mathfrak{B}f^{-1}(y)$ of the inverse image $f^{-1}(y)$ is compact for every point y of Y. A problem raised by Balachandran [1] will also be solved.

1. We shall prove

Lemma 1. *Let f be a closed continuous mapping of a normal T_1-space X onto a topological space Y. If Y satisfies the first countability axiom, then $\mathfrak{B}f^{-1}(y)$ is countably compact for every point y of Y.*

Proof. Let y be any point of Y. By the first countability axiom, there exists a countable collection $\{V_i \mid i=1,2,\cdots\}$ of open neighborhoods of y such that for any open neighborhood U there can be found some V_i with $V_i \subset U$.

Suppose that $\mathfrak{B}f^{-1}(y)$ is not countably compact. Then there exist a countable number of points x_i, $i=1,2,\cdots$ of $\mathfrak{B}f^{-1}(y)$ such that $\{x_i\}$ has no limit point. Then by the normality of X we can find a discrete collection $\{G_n\}$ of open sets of X such that

$$x_i \in G_i \quad \text{for } i=1,2,\cdots; \ G_i \cap G_j = 0 \quad \text{for } i \neq j$$

and $\{G_n\}$ is locally finite.

Since each point x_i belongs to the boundary $\mathfrak{B}f^{-1}(y)$ of $f^{-1}(y)$, there exists a point x_i' of X such that

$$x_i' \notin f^{-1}(y), \quad x_i' \in G_i \cap f^{-1}(V_i).$$

Then $\{x_i' \mid i=1,2,\cdots\}$ is locally finite in X and hence the set C consisting of all points x_i', $i=1,2,\cdots$ is closed. Therefore if we put $H=Y-f(C)$, H is an open set of Y. Since $x_i' \notin f^{-1}(y)$, we have $y \in H$. Hence there exists some V_i such that $V_i \subset H$. This implies that we have $f(x_i') \notin V_i$ for some i. On the other hand we have chosen the point x_i' so that $x_i' \in f^{-1}(V_i)$. This is a contradiction. Thus Lemma 1 is proved.