26. *Capacity of Subsets of the Ideal Boundary*

By Zenjiro KURAMOCHI
Mathematical Institute, Osaka University
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In the previous paper,[1] we introduced the notion of the capacity of the subset of the ideal boundary and proved some theorems. Unfortunately their proofs were much complicated. The purpose of the present article is to give simple proofs. Let R be a Riemann surface with a positive boundary. Let $\{R_n\}$ $(n=0,1,2,\cdots)$ be an exhaustion of R with compact relative boundaries $\{\partial R_n\}$[2] and D be a non compact domain in R whose relative boundary ∂D is composed of at most enumerably infinite number of analytic curves clustering nowhere in R. We say that a sequence $\{D\cap(R-R_n)\}$ determines a subset B_D of the ideal boundary.

1. **Capacity of a Subset B_D.** Let $U_{n,n+i}(z)$ be a harmonic function in $R_{n+i}-R_0-(D\cap(R_{n+i}-R_n))$ (in short we denote it by $B_{n,n+i}$) such that $U_{n,n+i}(z)=0$ on ∂R_0, $U_{n,n+i}(z)=1$ on $(\partial R_n\cap D)+(\partial D\cap(R_{n+i}-R_n))$ and $\dfrac{\partial U_{n,n+i}(z)}{\partial n}=0$ on $\partial R_{n+i}-D$. Then we have the Dirichlet's integral

$$\underset{B_{n,n+i}}{D}(U_{n,n+i}(z)-U_{n,n+i+j}(z),\ U_{n,n+i}(z))=0,$$

whence

$$\underset{B_{n,n+i}}{D}(U_{n,n+i+j}(z))=\underset{B_{n,n+i}}{D}(U_{n,n+i}(z))+\underset{B_{n,n+i}}{D}(U_{n,n+i}(z)-U_{n,n+i+j}(z)). \quad (1)$$

But it is easily seen by Dirichlet's principle that $\underset{B_{n,n+i}}{D}(U_{n,n+i}(z))\leqq \underset{R_1-R_0}{D}(U^*(z))\leqq M<\infty$ for every n and i, where $U^*(z)$ is a harmonic function in R_1-R_0 such that $U^*(z)=0$ on ∂R_0 and $U^*(z)=1$ on ∂R_1. Therefore by (1)

$$M\geqq \underset{B_{n,n+i+j}}{D}(U_{n,n+i+j}(z))\geqq \underset{B_{n,n+i}}{D}(U_{n,n+i+j}(z))\geqq \underset{B_{n,n+i}}{D}(U_{n,n+i}(z)),$$

hence the sequence $\{\underset{B_{n,n+i+j}}{D}(U_{n,n+i+j}(z))\}$ is convergent, which implies $\lim\limits_{\substack{i=\infty\\ j=\infty}}\underset{B_{n,n+i+j}}{D}(U_{n,n+i+j}(z)-U_{n,n+i}(z))=\lim\limits_{\substack{i=\infty\\ j=\infty}}\{\underset{B_{n,n+i+j}}{D}(U_{n,n+i+j}(z))-\underset{B_{n,n+i}}{D}(U_{n,n+i}(z))\}=0.$ Thus $\{U_{n,n+i}(z)\}$ converges to a function $U_n(z)$ in mean. Since $U_{n,n+i}(z)=0$ on ∂R_0, it converges uniformly in every compact set of $R-(D\cap(R-R_n))$. We see $U_{n+i,n+i+j}(z)\leqq U_{n,n+i+j}(z)$, by the maximum principle. From this we have $\lim\limits_{j=\infty}U_{n+i,n+i+j}(z)=U_{n+i}(z)\leqq U_n(z)=\lim\limits_{j=\infty}$

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1) Z. Kuramochi: Harmonic measures and capacity of sets of the ideal boundary. I–II, Proc. Japan Acad., **30–31** (1954–1955).

2) In this article, we denote by ∂A the relative boundary of A.