25. An Estimation of the Measure of Linear Sets

By Zenjiro KURAMOCHI
Mathematical Institute, Osaka University

(Comm. by K. KUNUGI, M.J.A., Feb. 13, 1956)

Let R be an abstract Riemann surface and suppose that a conformal metric is given on R, of which a line element ds is given by the local parameter t such that $ds=|\lambda(t)|dt$ and let O be a fixed point of R. Denote by D_ρ the domain bounded by the points having the distance ρ from O and suppose for $\rho<\infty$ that the domain D_ρ is compact and $\bigcup_{\rho>0} D_\rho=R$. The boundary ∂D_ρ of D_ρ is composed of $n(\rho)$ components, $r_1, r_2, \cdots, r_n$. Denote by $\Lambda(\rho)$ the largest length of r_k $(k=1, 2, \cdots, n(\rho))$, that is,

$$l_k=\int_{r_k} ds, \qquad \Lambda(\rho)=\max_k l_k.$$

Put $N(\rho)=\max_{\rho'<\rho} n(\rho)$. *A. Pfluger proved that*

if
$$\limsup_{\rho=\infty}\left[4\pi\int_{\rho_0}^{\rho}\frac{d\rho}{\Lambda(\rho)}-\log N(\rho)\right]=\infty,^{1)}$$

then

$$R\in O_{AB}.$$

The condition of this theorem depends not only the minimum modulus but also on the number of components. In this article we give a condition depending only on the minimum modulus but our criterion is applicable only to a special type of Riemann surface, i.e. the Riemann surface which is planer and whose boundary is a closed set on a straight line. Let $\{R_n\}$ $(n=1, 2, \cdots)$ be the exhaustion of R with compact relative boundaries $\{\partial R_n\}$. The open set $R_{n+1}-R_n$ $(n\geqq 1)$ consists of a finite number of ring domains $G_{i_1,i_2,\cdots,i_n}$ $(i_1=1, 2, \cdots, j_1,\ i_2=1, 2, \cdots, j_2, \cdots,\ i_n=1, 2, \cdots, j_n)$. Let $\omega(z)$ be a harmonic function in $G_{i_1,i_2,\cdots,i_n}$ such that $\omega(z)=0$ on the outer boundary of $G_{i_1,i_2,\cdots,i_n}$ contained in ∂R_n and $\omega(z)=1$ on the inner boundary of $G_{i_1,i_2,\cdots,i_n}$ contained in ∂R_{n+1}. Let $D(\omega(z))$ be the Dirichlet's integral of $\omega(z)$ and put $\mathrm{mod}\,(G_{i_1,i_2,\cdots,i_n})=1/D(\omega(z))$. We call it the modulus of $G_{i_1,i_2,\cdots,i_n}$ and further put $\mathfrak{M}_n=\min_{i_n}\mathrm{mod}\,(G_{i_1,i_2,\cdots,i_n})$. Then we can prove the following

Theorem. Let R be a planer domain and suppose that its ideal

---

1) A. Pfluger: Sur l'existence de fonctions non constants, analytiques, uniformes et borneés sur une surface de Riemann ouverte, C. R. Acad. Sci. Paris, 230 (1950).