24. *Uniform Convergence of Fourier Series. VI*

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6. Furthermore we can improve Theorem 6, in the following form:

Theorem 7. *If*

$$(1)\qquad \int_0^{|h|}(f(x+u)-f(x))du=o(|h|),\ \ as\ h\to 0$$

for a fixed x, and

$$(2)\qquad \frac{1}{h}\int_0^h(f(t+u)-f(t-u))du=o\Big(1\Big/\log\frac{1}{h}\Big),\ \ as\ h\to 0$$

uniformly for all t, then the Fourier series of $f(t)$ converges at x.

In other words the condition in Theorem 6

$$\int_0^{|h|}|f(x+u)-f(x)|du=o(|h|)$$

is replaced by (1).

Proof. We put

$$s_n(x)-f(x)=\frac{1}{\pi}\int_0^\pi \varphi_x(t)\frac{\sin nt}{t}dt+o(1)=\frac{1}{\pi}\Big[\int_0^{\pi/n}+\int_{\pi/n}^\pi\Big]+o(1)$$
$$=\frac{1}{\pi}[I+J]+o(1).$$

Then by integration by parts

$$I=\int_0^{\pi/n}\Phi_x(t)\Big(\frac{\sin nt}{t^2}-\frac{n\cos nt}{t}\Big)dt,$$

and hence, on account of (2), the absolute value of I is not greater than

$$2n\int_0^{\pi/n}|\Phi_x(t)|\frac{dt}{t}=o\Big(n\int_0^{\pi/n}dt\Big)=o(1),$$

where $\Phi_x(t)=\int_0^t\varphi_x(u)du=o(t)$ as $n\to\infty$ $(0\leqq t\leqq \pi/n)$.

In order to evaluate J we now put (cf. [4])

$$J=\int_{\pi/n}^\pi \varphi_x(t)\frac{\sin nt}{t}dt=J_1-J_2,$$

where

$$J_1=\sum_{k=1}^{(n-1)/2}\int_0^{\pi/n}\frac{\varphi_x(t+2k\pi/n)-\varphi_x(t+(2k-1)\pi/n)}{t+2k\pi/n}\sin nt\,dt,$$

$$J_2=\sum_{k=1}^{(n-1)/2}\int_0^{\pi/n}\varphi_x(t+(2k-1)\pi/n)\Big(\frac{1}{t+2k\pi/n}-\frac{1}{t+(2k-1)\pi/n}\Big)\sin nt\,dt,$$