23. *Note on the Mean Value of V(f). III*

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1. Let $GF(q)$ denote a finite field of order $q=p^\nu$ and put

(1.1) $$f(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_1x \qquad (a_j \in GF(q)),$$

where $1<n<p$. Let $V(f)$ denote the number of distinct values assumed by $f(x)$, $x \in GF(q)$. It is known [1] that

(1.2) $$\sum_{\deg f=n} V(f)=c_nq^n+O(q^{n-1}),$$

where the summation on the left-hand side is over all polynomials of degree n of the form (1.1) and

$$c_n=1-\frac{1}{2!}+\frac{1}{3!}-\cdots+(-1)^{n-1}\frac{1}{n!}.$$

In other words, the mean value of $V(f)$ over all polynomials f of degree n is asymptotically equal to c_nq.

Professor Carlitz has proposed, in a written communication to the author, a problem to evaluate the sum

$$\sum_{\deg f=n} V^2(f).$$

Here we wish to present a solution of this problem by proving the following

Theorem. *Under the Riemann hypothesis for L-functions we have*

(1.3) $$\sum_{\deg f=n} V^2(f)=c_n^2q^{n+1}+O(q^n),$$

where the summation on the left-hand side is extended over all polynomials of degree n of the form (1.1).

Thus the variance $q^{-n+1}\sum_{\deg f=n}(V(f)-c_nq)^2$ *is of order* $O(q)$.

The L-functions mentioned here were introduced and employed in [3] with certain characters defined over the polynomial ring $GF[q, x]$. For the effect of the Riemann hypothesis, see [3, Proposition 3].

2. Following the notation of [2, §3] we write

$$\lambda=\lambda^{(1)}\lambda^{(2)}\cdots\lambda^{(n-1)}$$

and put

$$\tau_j(\lambda)=\sum_{\deg M=j}\lambda(M),$$

the summation being over the primary polynomials in $GF[q, x]$ of degree j. Then, we have, as before,

$$\tau_j(\lambda_0)=q^j,$$