22. *Some Trigonometrical Series. XX*

By Shin-ichi IZUMI
Mathematical Institute, Tokyo Metropolitan University, Tokyo

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1. S. Chowla [1] proposed the problem (the cosine problem):

Let K be an arbitrary positive number. Then can we find $N=N(K)$ such that

$$(1) \qquad \min_{0\leqq x<2\pi} (\cos m_1x+\cos m_2x+\cdots+\cos m_nx)<-K$$

for all $n\geqq N$ where $(m_1, m_2,\cdots, m_n)$ is any set of n different positive integers?

We shall give an answer to this problem in a special case and prove theorems closely connected.

2. **Theorem 1.** *For any positive number K, there is an N such that (1) holds for all $n\geqq N$ and for any set of distinct integers $(m_1, m_2,\cdots, m_n)$ such that the number of solutions of the equations*

$$(2) \qquad m_i+m_j=m_k \qquad (i<j<k\leqq n)$$

is of $o(n^2)$.

Proof. If not so, there is a positive number K such that for any N there are an $n\geqq N$ and a set of distinct integers $(m_1, m_2, \cdots, m_n)$ such that

$$-f(x)=\sum_{k=1}^{n}\cos m_kx\geqq -K.$$

Then we have

$$-n\leqq f(x)\leqq K.$$

Let us consider the integral

$$I=\int_0^{2\pi}(K-f)^2(n+f)dx$$

which was used by S. Szidon [2] for a different purpose. By the Schwarz inequality

$$(3) \qquad I^2=\Big(\int_0^{2\pi}(K-f)^{3/2}\cdot(K-f)^{1/2}(n+f)dx\Big)^2$$

$$\leqq \int_0^{2\pi}(K-f)^3dx\int_0^{2\pi}(K-f)(f+n)^2dx=I_1\cdot I_2,$$

say. Now since

$$\int_0^{2\pi}f\,dx=0,\quad \int_0^{2\pi}f^2dx=\pi n^2,$$

we have

$$I=\int_0^{2\pi}(K^2-2Kf+f^2)(n+f)dx$$