20. *Some Strong Summability of Fourier Series*

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1. The object of this paper is to find the condition of almost everywhere convergence of the series

$$\sum_{n=1}^{\infty} |s_n(x)-f(x)|^k, \tag{1.1}$$

where $s_n(x)$ is the nth partial sum of the Fourier series of $f(x)$.

Concerning this problem, S. Izumi [2] has shown the following:

Let $p>1$, $p\geqq k>1$ and ε be any positive number. If

$$\omega_p(t)=\sup_{|u|\leqq t}\left\{\int_{-\pi}^{\pi}|f(x+u)-f(x)|^p dx\right\}^{1/p}\leqq A\, t^{1/k}\Big/\left(\log\frac{1}{t}\right)^{(1+\varepsilon)/k},$$

then the series (1.1) converges almost everywhere.

Related this theorem we shall prove some theorems.

Theorem 1. *In order that the series (1.1) converges almost everywhere, one of the following conditions is sufficient:*

$$\sum_{\lambda=1}^{\infty}\lambda^{\gamma}[2^{\lambda/k}\omega_p(1/2^{\lambda})]^p<\infty,\quad \textit{for}\ 2\geqq p>k>1,\ \gamma>p/k-1, \tag{1.2}$$

$$\sum_{\lambda=1}^{\infty}2^{\lambda}[\omega_p(1/2^{\lambda})]^p<\infty,\qquad \textit{for}\ 2>p=k>1, \tag{1.3}$$

$$\sum_{\lambda=1}^{\infty}\lambda\cdot 2^{\lambda}[\omega_p(1/2^{\lambda})]^p<\infty,\quad \textit{for}\ p=k=2. \tag{1.4}$$

2. Proof of Theorem 1.[1] We have

$$\begin{aligned} s_n(x)-f(x)&=\frac{1}{\pi}\int_0^{\pi}\varphi_x(t)\sin(n+1/2)t/(2\sin t/2)\,dt\\ &=\frac{1}{\pi}\int_0^{\pi}\varphi_x(t)\frac{\cos t/2}{2\sin t/2}\sin nt\,dt+\frac{1}{\pi}\int_0^{\pi}\frac{1}{2}\varphi_x(t)\cos nt\,dt\\ &=P_n(x)+Q_n(x), \end{aligned}$$

say, where $\varphi_x(t)=\varphi(t)=f(x+t)+f(x-t)-2f(x)$, and $P_n(x)$ and $Q_n(x)$ are the nth Fourier coefficients of the functions $\varphi_x(t)\cos t/2/(2\sin t/2)$ $=\varphi_x(t)p(t)$ and $\varphi_x(t)/2$, respectively.

Let $1<p\leqq 2$ and p' be its conjugate, then by the Hausdorff-Young inequality, we get[2]

$$\left\{\sum_{n=1}^{\infty}|P_n(x)\sin nh|^{p'}\right\}^{p/p'}\leqq A\left\{\int_0^{\pi}|\varphi(t+h)p(t+h)-\varphi(t-h)p(t-h)|^p dt\right\}$$

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1) Cf. N. Matsuyama [3].

2) We denote by A an absolute constant, which is not necessarily the same in different occurrences.