40. Contribution to the Theory of Semi-groups. I

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Following S. Schwarz [5], a semi-group*) S is called a *periodic semi-group* if, for every element a of S, the semi-group $(a)=\{a \mid a, a^2, \cdots, a^n \cdots\}$ generated by a contains a finite number of different elements.

It is known that the commutative finite semi-group (a) for every $a \in S$ contains only one idempotent (for detail, see K. Iséki [3]). Let e be an idempotent, and $K^{(e)}$ the set of all elements a such that (a) contains the idempotent e, i.e. $K^{(e)}=\{a \mid a^\rho=e \text{ for some } \rho\}$.

S. Schwarz [5] proved that if S is *commutative* or *totally non-commutative*, $K^{(e)}$ is a maximal semi-group belonging to the idempotent e of S and S is the sum of disjoint semi-groups $K^{(e)}$, each containing only one idempotent.

G. Thierrin [7] defined a new class of semi-groups as follows: A semi-group is said to be *strongly reversible*, if, for any two elements a, b of S, there are three positive integers r, s, and t such that

$$(ab)^r=a^s b^t=b^t a^s.$$

It is clear that any commutative semi-group is strongly reversible. We shall show Theorem 1 which is a generalisation of S. Schwarz result.

Theorem 1. If a periodic semi-group S is strongly reversible, then $K^{(e)}$ is a semi-group.

Proof. Let a, b be any two elements of $K^{(e)}$, then there are integers ρ, τ such that $a^\rho=e=b^\tau$. Since S is strongly reversible, there are three integers r, s, and t such that

$$(ab)^r=a^s b^t=b^t a^s.$$

Therefore, we have

$$\begin{aligned}(ab)^{r\rho\tau}&=((ab)^r)^{\rho\tau}=(a^s b^t)^{\rho\tau}\\&=a^{s\rho\tau}b^{t\rho\tau}=e\cdot e=e.\end{aligned}$$

Hence $ab \in K^{(e)}$. This completes the proof.

Theorem 1 and a result of S. Schwarz [5] imply the following

Theorem 2. Any strongly reversible periodic semi-group is the sum of disjoint semi-groups, each containing only one idempotent.

Let S be a strongly reversible or a totally non-commutative periodic semi-group. Then we shall remark that *each $K^{(e)}$ does not*

*) For general theory of semi-groups, see P. Dubreil [1].