38. *A Theorem of Dimension Theory*

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Recently a dimension theory for general metric spaces has been established by M. Katětov and K. Morita.[1] The purpose of this note is to study some necessary and sufficient conditions for n-dimensionality of general metric spaces. In the present note we take the definition of dimension by H. Lebesgue or that by M. Katětov and K. Morita as the same: $\dim R=-1$ for $R=\phi$, $\dim R\leqq n$ if and only if for any pair of a closed set F and an open set G with $F\subseteq G$ there exists an open set U such that $F\subseteq U\subseteq G$, $\dim B(U)\leqq n-1$.[2]

Definition. For two collections $\mathfrak{U},\mathfrak{U}'$ of open sets we denote by $\mathfrak{U}<\mathfrak{U}'$ the fact that $U\subseteq U'$ for every $U\in\mathfrak{U}$ and for some $U'\in\mathfrak{U}'$.

Definition. We mean by a *disjoint collection* a collection $\mathfrak{U}$ of open sets such that $U, U'\in\mathfrak{U}$ and $U\neq U'$ imply $U\frown U'=\phi$.

Theorem 1. *In order that* $\dim R\leqq n$ *for a metric space* R *it is necessary and sufficient that there exist* $n+1$ *sequences* $\mathfrak{U}_1^i>\mathfrak{U}_2^i>\cdots$ $(i=1,2,\cdots,n+1)$ *of disjoint collections such that* $\{\mathfrak{U}_m^i\mid i=1,\cdots,n+1; m=1,2,\cdots\}$ *is an open basis of* R.

Proof. If $\dim R=0$,[3] then from M there exists a sequence $\mathfrak{V}_m$ $(m=1,2,\cdots)$ of locally finite coverings consisting of open, closed sets such that $S(p,\mathfrak{V}_m)$ $(m=1,2,\cdots)$[4] is a nbd (=neighbourhood) basis of each point p of R. For $\mathfrak{V}_m=\{V_\alpha\mid\alpha<\tau\}$ we define $\mathfrak{V}'_m=\{V_\alpha-\smile_{\beta<\alpha}V_\beta\mid\alpha<\tau\}$ and $\mathfrak{U}_1=\mathfrak{V}'_1$, $\mathfrak{U}_2=\mathfrak{U}_1\wedge\mathfrak{V}'_2$, $\mathfrak{U}_3=\mathfrak{U}_2\wedge\mathfrak{V}'_3,\cdots$. Then $\mathfrak{U}_1>\mathfrak{U}_2>\cdots$ is a sequence of disjoint collections, and $\{\mathfrak{U}_m\mid m=1,2,\cdots\}$ is an open basis of R.

Conversely, if there exists a sequence $\mathfrak{U}_1>\mathfrak{U}_2>\cdots$ of disjoint

1) M. Katětov: On the dimension of non-separable spaces. I, Czechoslovak Mathematical Journal, **2** (77), (1952). K. Morita: Normal families and dimension theory for metric spaces, Math. Annalen, **128** (1954); A condition for the metrizability of topological spaces and for n-dimensionality, Science Reports of the Tokyo Kyoiku Daigaku, Sect. A, **5**, No. 114 (1955).

2) $B(U)$ denotes the boundary of U. See K. Morita: Normal families and dimension theory for metric spaces; from now forth we call this paper M.

3) From now forth we assume $R\neq\phi$.

4) In this note we concern ourselves only with open coverings. We call $\mathfrak{V}$ a locally finite covering if every point of R has some neighbourhood intersecting only finitely many elements of $\mathfrak{V}$. $S(A,\mathfrak{V})=\smile\{V\mid V\in\mathfrak{V}, V\frown A\neq\phi\}$ for $A\subseteq R$. Notations of this paper are chiefly due to J. W. Tukey: Convergence and uniformity in topology (1940).