59. A Generalisation of Wallace Theorem on Semi-groups

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In his paper [4], A. D. Wallace has proved the following

Theorem 1. If S is a compact abelian semi-group and if each element of S is idempotent, then S has a zero-element.

In this Note, we shall extend his result to homogroup, a larger class of semi-groups.

A homogroup was studied by G. Thierrin, A. H. Clifford and D. D. Miller [1]. Following G. Thierrin [3], we shall define homogroups.

Definition. A semi-group S is called *homogroup*, if

(1) S contains an idempotent e.

(2) For each $x \in S$, there are elements x' and x'' such that $xx'=e=x''x$.

(3) For any $x \in S$, $xe=ex$.

G. Thierrin [3] proved that $N=\{xe \mid x \in S\}$ is a group and a two-sided ideal. It is clear that the idempotent e is the unit of N.

Now, we shall prove the following theorem which is a generalisation of A. D. Wallace's result [4].

Theorem 2. If each element of a homogroup S is idempotent, then S has a zero-element.

Proof. Let x be an element of S, then the element xe is an idempotent, by the assumption of S and $xe \in N$. Hence

$$xe \cdot xe = xe.$$

Since N is a group, xe has an inverse in N. Therefore, for every $x \in S$, we have

$$xe = e.$$

This shows that e is a zero-element of S. The proof is complete.

It is known that *any compact abelian semi-group is a homogroup* (see K. Iséki [2]). Therefore, if each element of a compact abelian semi-group S is idempotent, then, by Theorem 2, S has a zero-element. Thus, the proof of Theorem 1 is complete.

References

[1] A. H. Clifford and D. D. Miller: Semi-groups having zeroid elements, Amer. Jour. Math., **70**, 117–125 (1948).

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[3] G. Thierrin: Sur les homogroupes, C. R. Acad. Sci., Paris, **234**, 1519–1521 (1952).

[4] A. D. Wallace: A note on mobs, Anais Acad. Brasil. Ciencias, **24**, 329–334 (1952).