55. On a Relation between Dimension and Metrization

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The purpose of the present note is to show that the dimension of a metrizable space is defined by some characters of metrics which agree with the topology of the space.

In this note we concern ourselves only with metrizable spaces and take the definition of dimension by H. Lebesgue or the equivalent definition: $\dim \phi = -1$, $\dim R \leqq n$ if for any pair of a closed subset F and an open subset G with $F \subseteq G$ there exists an open set U such that $F \subseteq U \subseteq G$, $\dim(\overline{U} - U) \leqq n-1$.[1]

We state here a theorem previously proved by the author[2] which will be needed in the proof of our main theorem.

In order that a T_1 topological space R is a metrizable space with $\dim R \leqq n$ it is necessary and sufficient that there exists a sequence $\mathfrak{V}_1 > \mathfrak{V}_2^* > \mathfrak{V}_2 > \mathfrak{V}_3^* > \cdots$ of open coverings such that $S(p, \mathfrak{V}_m)$ $(m=1, 2, \cdots)$[3] is a nbd (=neighbourhood) basis for each point p of R and such that each set of $\mathfrak{V}_{m+1}$ intersects at most $n+1$ sets of $\mathfrak{V}_m$.

Theorem. *In order that $\dim R \leqq n$ for a metrizable space R it is necessary and sufficient that one can assign a metric $\rho(x, y)$ agreeing with the topology of R such that for every $\varepsilon > 0$ and for every point x of R, $\rho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ $(i=1, \cdots, n+2)$ imply $\rho(y_i, y_j) < \varepsilon$ for some i, j with $i \neq j$.*[4]

Proof. Necessity. 1. Let R be a metrizable space with $\dim R \leqq n$, then from the above stated theorem there exists a sequence $\mathfrak{U}_1 > \mathfrak{U}_2^{**} > \mathfrak{U}_2 > \mathfrak{U}_3^{**} > \cdots$[5] of open coverings of R such that $S(p, \mathfrak{U}_m)$ $(m=1, 2, \cdots)$ is a nbd basis for each point p of R and such that each $S^2(p, \mathfrak{U}_{m+1}^*)$ intersects at most $n+1$ sets of $\mathfrak{U}_m$. Now, we define $S_{m_2, m_3, \cdots, m_p}(U)$ for $1 \leqq m_1 < m_2 < \cdots < m_p$ and for $U \in \mathfrak{U}_{m_1}$ such that $S_{m_2}(U) = \cup \{U' | S(U', \mathfrak{U}_{m_2}) \cap U \neq \phi, U' \in \mathfrak{U}_{m_2}\} = S^2(U, \mathfrak{U}_{m_2})$, $S_{m_2, \cdots, m_p}(U) = \cup \{U' | S(U', \mathfrak{U}_{m_p}) \cap S_{m_2, \cdots, m_{p-1}}(U) \neq \phi, U' \in \mathfrak{U}_{m_p}\} = S^2(S_{m_2, \cdots, m_{p-1}}(U), \mathfrak{U}_{m_p})$ and $S_{m_2, \cdots, m_p}(U) = U$ for $p=1$. Furthermore we define $\mathfrak{S}_{m_1} = \mathfrak{U}_{m_1}$,

1) The equivalence of these two definitions was proved by M. Katětov: On the dimension of non-separable spaces I, Czechoslovak Mathematical Journal, **2** (77) (1952), and by K. Morita: Normal families and dimension theory for metric spaces, Math. Annalen, **128** (1954), independently.

2) A theorem of dimension theory, Proc. Japan Acad., **32**, No. 3 (1956).

3) $S(p, \mathfrak{V}) = \cup \{V | p \in V \in \mathfrak{V}\}$, $S(A, \mathfrak{V}) = \cup \{V | V \in \mathfrak{V}, A \cap V \neq \phi\}$ for a subset A and a covering $\mathfrak{V}$.

4) $S_{\varepsilon/2}(x) = \{y | \rho(x, y) < \varepsilon/2\}$, $\rho(S_{\varepsilon/2}(x), y_i) = \inf\{\rho(z, y_i) | z \in S_{\varepsilon/2}(x)\}$.

5) $\mathfrak{U}^* = \{S(U, \mathfrak{U}) | U \in \mathfrak{U}\}$, $\mathfrak{U}^{**} = (\mathfrak{U}^*)^*$, $S^2(p, \mathfrak{U}) = S(S(p, \mathfrak{U}), \mathfrak{U})$.