53. *Evans-Selberg's Theorem on Abstract Riemann Surfaces with Positive Boundaries. I*

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Let R^* be a Riemann surface with a positive boundary and let $\{R_n\}$ $(n=0,1,2,\cdots)$ be its exhaustion with compact relative boundaries $\{\partial R_n\}$. Put $R=R^*-R_0$. Let $N(z,p)$ be a positive function in R harmonic in R except one point $p\in R$ such that $N(z,p)=0$ on ∂R_0, $N(z,p)+\log|z-p|$ is harmonic in a neighbourhood of p and the $*$-Dirichlet integral taken over R is minimal, where the $*$-Dirichlet integral is taken with respect to $N(z,p)+\log|z-p|$ in a neighbourhood of p. It is easily seen that such $N(z,p)$ is uniquely determined and $\int_{\partial R_0}\frac{\partial N(z,p)}{\partial n}ds=2\pi$. As in the case when R^* is a Riemann surface with a null-boundary, we define the ideal boundary point, by making use of $N(z,p)$, that is, if $\{p_i\}$ is a sequence of points in R having no accumulation point in $R+\partial R_0$, for which the corresponding functions $N(z,p_i)$ $(i=1,2,\cdots)$ converge uniformly in every compact set of R, we say that $\{p_i\}$ is a fundamental sequence. Two fundamental sequences are equivalent, if and only if, their corresponding sequences of functions have the same limit function. The equivalent sequences are made to correspond to an ideal boundary point. The set of all the ideal boundary points will be denoted by B and the set $R+B$, by $\overline{R}$. The domain of definition of $N(z,p)$ may now be extended by writing $N(z,p)=\lim_{i=\infty}N(z,p_i)$ $(z\in R,\ p\in B)$, where $\{p_i\}$ is any fundamental sequence. For p in B, the flux of $N(z,p)$ along ∂R_0 is also 2π. The distance between two points p_1 and p_2 of $\overline{R}$ is defined by

$$\delta(p_1,p_2)=\sup_{z\in R_1-R_0}\left|\frac{N(z,p_1)}{1+N(z,p_1)}-\frac{N(z,p_2)}{1+N(z,p_2)}\right|.$$

The topology induced by this metric is homeomorphic to the original topology in R and we see easily that $R-R_1+\partial R_1+B$ and B are closed and compact.

At first, we have the following

Lemma 1. Put $N^M(z,p)=\min[M,N(z,p)]$. *Then the Dirichlet integral of* $N^M(z,p)$ *over* R *satisfies*

$$D(N^M(z,p))\leqq 2\pi M,\qquad M\geqq 0,$$

for every point of $\overline{R}$.

In what follows, in order to introduce the harmonicity or superharmonicity in $\overline{R}$ (not only in R), we make some preparations as follows.