# 52. Contribution to the Theory of Semi-groups. II

By Kiyoshi ISÉKI

Kobe University

(Comm. by K. KUNUGI, M.J.A., April 12, 1956)

Any compact semi-group contains at least one idempotent. This theorem has been proved by some writers (cf. K. Iséki [2], Th. 3).

Let $E$ be the set of all idempotents $e_\alpha$ of a given compact semi-group $S$, then $E$ is non-empty.

If $e_\alpha e_\beta = e_\alpha$ for $e_\alpha$, $e_\beta \in E$, we shall write $e_\alpha \leq e_\beta$. The order relation $\leq$ defines a quasi-order on $E$. If $E$ is commutative, then $E$ is a partial order set relative to the order.

In this Note, we shall first extend a result of S. Schwarz [3].

We shall first prove that

$$\mathfrak{N} = \bigcap_{e_\alpha \in E} Se_\alpha S$$

is non-empty. By the compactness of $S$, each $Se_\alpha S$ is closed. For any finite $e_{\alpha_1}, e_{\alpha_2}, \cdots, e_{\alpha_k}$, we have

$$e_{\alpha_1} \cdot e_{\alpha_2} \cdots e_{\alpha_k} \in Se_{\alpha_1}S \cdot Se_{\alpha_2}S \cdots Se_{\alpha_k}S$$
$$\subseteq Se_{\alpha_1}S \frown Se_{\alpha_2}S \frown \cdots \frown Se_{\alpha_k}S.$$

Therefore, $Se_{\alpha_1}S \frown Se_{\alpha_2}S \frown \cdots \frown Se_{\alpha_k}S$ is non-empty, and $\mathfrak{N}$ is non-empty. It is clear that $\mathfrak{N}$ is a closed two-sided ideal, and hence $\mathfrak{N}$ is a compact semi-group. For $a \in \mathfrak{N}$, $SaS$ is a closed ideal of $\mathfrak{N}$. The compact semi-group $SaS$ contains an idempotent $e$. Therefore $SeS \subseteq SaS \subseteq \mathfrak{N} \subseteq SeS$. Hence $SaS = SeS = \mathfrak{N}$, for any $a$ and any idempotent $e$ of $\mathfrak{N}$. $\mathfrak{N} = SaS$ is a closed minimal two-sided ideal.

Thus, this fact shows that *there is a closed minimal two-sided ideal in $S$.*

If $S$ is a compact homogroup in the sense of G. Thierrin [4], then $S$ contains a compact group and two-sided ideal $m$ of $S$. Therefore, $\mathfrak{N} \subset m$. As any group does not contain proper ideal, $\mathfrak{N} = m$. Therefore, $\mathfrak{N}$ is a compact group. Hence $\mathfrak{N}$ contains only one idempotent $e$, which is the unit element of $\mathfrak{N}$. Let $e'$ be an idempotent of $S$, then, by the definition of $\mathfrak{N}$, $\mathfrak{N} \subseteq \mathfrak{N}e'\mathfrak{N} \subseteq \mathfrak{N}$. Hence $ee'e \in \mathfrak{N}$. Since $S$ is a homogroup, $e$ is permutable with any element of $S$. Hence $ee'e = ee'$ and $ee'$ is an idempotent. Therefore $ee' = e$ and this shows $e \leq e'$. So we can state the following

*Theorem 1. Any compact homogroup has a unique least idempotent.**)

*) Theorem 1 is proved without the assumption of compactness.