76. On the σ-weak Topology of W^*-algebras

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1. Preliminaries. The author [5] had shown that the σ-weak topology of W^*-algebras is free from the adjoint operation as follows:

Theorem A. *Suppose that a C^*-algebra M is the adjoint space of a Banach space F, then it is a W^*-algebra and the topology $\sigma(M, F)$ of M is the σ-weak topology.*

This will suggest that the following question is affirmative: *Suppose that ϕ is an algebraic isomorphism (not necessarily adjoint preserving) of a W^*-algebra onto another. Then, can we conclude that ϕ is σ-weakly bicontinuous?*

The purpose of this paper is to prove this in a more general form (§2, Theorem 2).

2. Theorems. Let M be a C^*-algebra, M^* the adjoint space of M.

Definition. A subspace V of M^* is said invariant, if $f \in V$ implies f_a, ${}_bf \in V$ for any $a, b \in M$, where $f_a(x)=f(xa)$ and ${}_bf(x)=f(bx)$.

Theorem 1. *Let V be an invariant subspace of M^* which is everywhere $\sigma(M^*, M)$-dense in M^*, then $V\cap S$ is everywhere $\sigma(M^*, M)$-dense in S, where S is the unit sphere of M^*.*

Proof. Put $T_af=f_a$ for $f \in V$, then T_a is a linear operator on the normed space V and moreover $\| T_af\| = \sup_{|||x|||\leq 1} |f(xa)| \leq \|f\| \; |||a|||$; hence $\|T_a\| \leq |||a|||$, where $\|T_a\|$ is the operator norm of T_a.

Suppose that $T_a=0$, then $(T_af)(x)=f(xa)=0$ for all $f \in V$ and $x \in M$. Since V is everywhere $\sigma(M^*, M)$-dense in M^*, $xa=0$ for all $x \in M$; hence $a=0$. Moreover $T_{ab}=T_aT_b$ and so the mapping $a \to T_a$ is an isomorphism; hence by the minimality of C^*-norm [cf. [1], Th. 10] $\|T_a\|=|||a|||$ for all $a \in M$. Therefore,

$$|||a||| = \sup_{|||x|||\leq 1,\, f\in V\cap S} |f(xa)| = \sup_{|||x|||\leq 1,\, f\in V\cap S} |{}_xf(a)|$$
$$\leq \sup_{f\in V\cap S} |f(a)| \quad (\|{}_xf\| \leq |||x||| \; \|f\|),$$

so that $|||a|||=\sup_{f\in V\cap S}|f(a)|$ for all $a \in M$; hence the bipolar of $V\cap S$ in E^* is S, that is, $V\cap S$ is everywhere $\sigma(M^*, M)$-dense in S. This completes the proof.

J. Dixmier [2] had shown a characterization of adjoint Banach spaces as follows: Let E be a Banach space, E^* the adjoint space of E and V be a subspace which is strongly closed and everywhere