91. A Theorem on Paracompact Spaces

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Recently, K. Fujiwara [3] and K. Iséki [4] have shown that some properties on compact spaces are generalized very naturally to uniform spaces having Lebesgue property. In this paper, we shall extend a theorem of I. Gelfand and G. Silov [5] to paracompact space.

Let M be a metric space with metric ρ, and let $f(x)$ be a function defined on M. For a point $x\in M$, we shall define the *oscillation* of the function $f(x)$ at the point x. By ω (x, ε), we denote the least upper bound of ρ $(f(p), f(q))$ for $p, q\, S(x, \varepsilon)$, where $S(x, \varepsilon)$ is the sphere with center at x and radius ε. Then $\lim_{\varepsilon\to 0}\omega$ (x, ε) $(=\omega(x))$ exists and this limit is called the *oscillation of the function* $f(x)$ *at the point* x.

It is well known that *a function* $f(x)$ *defined on a metric space is continuous at a point* x, *if and only if the oscillation of* $f(x)$ *at* x *is equal to zero* (see W. Sierpiński [6], p. 184).

I. Gelfand and G. Silov [5] proved the following proposition. *Let* $\varphi(x)$ *be a function defined on a compact set* M *in n-dimensional Euclidean space* R^n, *and let* $\omega(x)\leq\varepsilon$ *for every point* $x\in M$, *then, there is a continuous function* $f(x)$ *on* M *such that* $|f(x)-\varphi(x)|\leq 2\varepsilon$.

We shall extend the proposition by I. Gelfand and G. Silov to more general topological space. First of all, suppose that M is a compact metric space. By the compactness of M, we can find a positive number η such that $\rho(x', x'')<\eta$ implies $|\varphi(x')-\varphi(x'')|\leq 2\varepsilon$. The open covering $\{S(x, \eta)\,|\,x\in M\}$ of M has a finite covering $S(x_1, \eta),\cdots,$ $S(x_n, \eta)$. Since M is a normal space, for the finite number of the open sets $S(x_i, \eta)$ $(i=1, 2,\cdots, n)$, there is such a decomposition $\lambda_i(x)$ $(i=1, 2,\cdots, n)$ of unity that

(1) each $\lambda_i(x)$ is a non-negative, continuous function on M,

(2) $1=\sum_{i=1}^{n}\lambda_i(x)$ for every x of M,

(3) $\lambda_i(x)=0$ on $M-S(x_i, \eta)$ $(i=1, 2,\cdots, n)$.

(See N. Bourbaki [2], p. 66.) To define a continuous function $f(x)$, let $f(x_i)=\varphi(x_i)$ and

$$f(x)=\sum_{i=1}^{n}\lambda_i(x)f(x_i),$$

then $f(x)$ is continuous on M. For any x of M, there is an open sphere $S(x_i, \eta)$ containing x_i.