90. Notes on Topological Spaces. IV. Function Semiring on Topological Spaces

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In our previous paper [3], we generalized some results on the theory of the space of maximal ideals of a semiring by W. Slowikowski and W. Zawadowski [4]. In this paper, we shall consider the relation between a function semiring on a normal space S and a lattice of closed sets in S. By using the result of it, we shall prove some theorems on function semiring. Such a consideration for function ring was also treated by G. Higman [2].

Let S be a T_2-space. Let $C^+(S)$ be the set of all continuous, bounded, non-negative real-valued functions on S, and let L be the lattice of all closed sets in S. $C^+(S)$ is a semiring*) with respect to the usual addition and multiplication and further $C^+(S)$ is a *positive semiring* in the sense of W. Slowikowski and W. Zawadowski [4].

We assume that we are familiar with the notions of proper ideals, maximal ideals of $C^+(S)$ and proper filter, ultrafilter of L (see, K. Iséki and Y. Miyanaga [3], and N. Bourbaki [1]). Following G. Higman [2], we shall first give a correspondence between ideals of $C^+(S)$ and filters of L.

Let I be an ideal of $C^+(S)$, then we shall define a set I^* of closed sets of S as follows: $A \in I^*$ if and only if, for any closed set F not meeting A, there is a function f of I such that the lower bound of f on F is positive, i.e. $\inf f(x) > 0$ on F.

Let J be a filter of L, and $f \in J^*$ if and only if, for every positive ε, there is a closed set A of J such that $f < \varepsilon$ on A.

Let I be a proper ideal of $C^+(S)$, and $A \in I^*$, then it is clear that $A \cup B \in I^*$ for every $B \in L$. Let A, $B \in I^*$, and let F be a closed set such that $F \frown (A \frown B) = 0$. Then $F \frown B$ does not meet A, therefore there is a function $f_1 \in I$ such that the lower bound α_1 of f_1 on $F \frown B$ is positive. For $\frac{1}{2}\alpha_1$, let $F_1 = \left\{x \mid f_1(x) \leq \frac{1}{2}\alpha_1\right\}$, then F is non-empty closed set and $F \frown F_1$ does not meet B. Therefore there is a function f_2 of L such that $\inf f_2 = \alpha_2$ is positive on $F \frown F_1$. Since I is an ideal, $f_1 + f_2 \in I$ and $f_1(x) + f_2(x) \geq \min\left(\frac{1}{2}\alpha_1, \alpha_2\right)$ on F. Hence I^* is a filter.

*) For the precise definition of semirings, see K. Iséki and Y. Miyanaga [3].