103. Fourier Series. I. Parseval Relation

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(Comm. by Z. SUETUNA, M.J.A., July 12, 1956)

1. In our former paper [1] we have proved the following theorems:

Theorem I. *If $g(t)$ is integrable and if $f(t)$ is a bounded measurable function such that*

$$\int_0^h (f(x+u)-f(x-u))\,du = o\left(h\Big/\log\frac{1}{h}\right)$$

uniformly for all x as $h\to 0$, then the Parseval relation

$$\frac{1}{\pi}\int_0^{2\pi} f(x)g(x)\,dx = \frac{a_0a_0'}{4}+\sum_{n=1}^{\infty}(a_na_n'+b_nb_n') \tag{1}$$

holds, where a_n, b_n and a_n', b_n' are Fourier coefficients of $f(t)$ and $g(t)$, respectively, and the right side series converges.

Theorem II. *If $g(t)$ is bounded measurable and if $f(t)$ is an integrable function such that*

$$\int_0^{2\pi} dx\left|\frac{1}{\delta}\int_0^{\delta}(f(x+u)-f(x-u))\,du\right| = o\left(1\Big/\log\frac{1}{\delta}\right),$$

then the Parseval relation (1) holds, where the right side series converges.

It is easy to see that if $f(t)$ and $g(t)$ belong to the conjugate classes and the Fourier series of one of them converges uniformly, then the Parseval relation (1) holds and the right side series converges in the ordinary sense.

Concerning the uniform convergence of Fourier series, the Salem theorem [3] (cf. [2]) is well known which reads as follows:

Theorem III. *If $f(t)$ is continuous and*

$$\sum_{k=1}^{[N/2]}\frac{f(t\pm 2k\pi/N)-f(t\pm(2k-1)\pi/N)}{k}$$

tends to zero uniformly as $N\to\infty$, then the Fourier series of $f(t)$ converges uniformly.

In this paper we show that if $f(t)$ and $g(t)$ belong to the conjugate classes, then the weaker one than the condition of Theorem III is sufficient for the validity of the Parseval relation (1).

2. **Theorem 1.** *If $g(t)$ is Lebesgue integrable and $f(t)$ is a bounded measurable function such that*

$$N\int_x^{x+\pi/N} dt\left|\sum_{k=1}^{[N/2]}\frac{f(t\pm(2k-1)\pi/N)-f(t\pm 2k\pi/N)}{k}\right| \tag{2}$$