132. Note on Dimension Theory

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(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1956)

Recently we have studied some necessary and sufficient conditions for n-dimensionality of general metric spaces.[1] The purpose of this note is to develop the previous results. That is, we shall give a generalization of our previous theorem concerning the relation between (Lebesgue's) dimension and covering and shall give some relations between metric function, length of covering and dimension. Moreover, we shall investigate embedding of n-dimensional metric spaces into products of 1-dimensional spaces on the foundation of our previous results.

All the topological spaces considered here are general metric spaces or metrizable spaces, and all the coverings are open unless otherwise mentioned.

Definition. A real valued function ρ of two points of a topological space R is a *non-Archimedean parametric* if

i) $\rho(x, y) \geqq 0$,

ii) $\rho(x, y) = \rho(y, x)$,

iii) $\{y \mid \rho(x, y) < \varepsilon\}$ is open for every $\varepsilon > 0$,

iv) $\rho(x, y) \leqq \max[\rho(x, z), \rho(y, z)]$.

Theorem 1. *In order that* $\dim R \leqq n$ *for a metrizable space* R *it is necessary and sufficient that one can assign a metric* $\rho(x, y)$ *agreeing with the topology of* R *such that* $\rho(x, y) = \inf\{\rho_0(x, z_1) + \rho_0(z_1, z_2) + \cdots + \rho_0(z_p, y) \mid z_i \in R\}$, $\rho_0(x, y) = \min\{\rho_i(x, y) \mid i = 1, \cdots, n+1\}$ *for some* $n+1$ *non-Archimedean parametrics* $\rho_i(x, y)$ $(i = 1, \cdots, n+1)$.[2]

Proof. Necessity. Let $\dim R \leqq n$, then there exist $n+1$ 0-dimensional subspaces R_i such that $R = \bigcup_{i=1}^{n+1} R_i$ from the general decomposition theorem due to M. Katětov and to K. Morita.[3] We assign a metric $\rho'(x, y)$ of R such that $\rho'(x, y) \leqq 1$. Since R_i $(i = 1, \cdots, n+1)$ are 0-dimensional, we get disjoint coverings[4] $\mathfrak{U}_m^i$ $(i = 1, \cdots, n+1, m = 1, 2 \cdots)$

1) A theorem of dimension theory, Proc. Japan Acad., **32**, No. 3 (1956). On a relation between dimension and metrization, Proc. Japan Acad., **32**, No. 4 (1956).

2) This theorem contains, as a special case for $n = 0$, Groot's theorem. See J. de Groot and H. de Vries: A note on non-Archimedean metrizations, Proceedings Koninkl. Nederl. Akademie van Wetenschappen, Ser. A, **58**, No. 2 (1955).

3) M. Katětov: On the dimension of non-separable spaces I, Czechoslovak Math. Jour., **2** (77) (1952). K. Morita: Normal families and dimension theory for metric spaces, Math. Annalen, **128** (1954).

4) We call a collection $\mathfrak{U}$ of open sets disjoint open collection if every intersection of two disjoint elements of $\mathfrak{U}$ is vacuous. If $\mathfrak{U}$ is a covering, it is called a disjoint covering. See "A theorem of dimension theory."