122. *Fourier Series. III. Wiener's Problem*

By Shin-ichi IZUMI
Department of Mathematics, Hokkaidô University, Sapporo
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1. N. Wiener [1] proposed to study the convergence of the series

$$\sum_{n=1}^{\infty} |s_n(x)-f(x)|^{\lambda}, \tag{1}$$

where $s_n(x)$ is the nth partial sum of the Fourier series of $f(x)$. In the former paper [2], we have proved the following

Theorem 1. *Let $p \geqq \lambda > 1$ and $\varepsilon > 0$. If*

$$\omega_p(t,f)=\max_{0<u<t}\left(\int_0^{2\pi} |f(x+u)-f(x)|^p dx\right)^{1/p} = O\left(\frac{t^{1/\lambda}}{(\log 1/t)^{(1+\varepsilon)/\lambda}}\right),$$

then the series (1) converges almost everywhere.

Further M. Kinukawa [3] proved the following

Theorem 2. *If one of the following conditions (a), (b), (c) is satisfied, then the series (1) converges almost everywhere:*

$$\text{(a)} \qquad \sum_{k=1}^{\infty} k^{\gamma}(2^{k/\lambda}\omega_p(1/2^k))^p < \infty \qquad (2 \geqq p > \lambda > 1,\ \gamma > p/\lambda - 1),$$

$$\text{(b)} \qquad \sum_{k=1}^{\infty} 2^k(\omega_p(1/2^k))^p < \infty \qquad (2 > p = \lambda > 1),$$

$$\text{(c)} \qquad \sum_{k=1}^{\infty} k2^k(\omega_p(1/2^k))^p < \infty \qquad (p = \lambda = 2).$$

If $1 < p \leqq 2$, Theorem 2 contains Theorem 1 as a particular case. We shall here prove the following

Theorem 3. *If*

$$\sum_{n=1}^{\infty} \omega_{\lambda}^{\lambda}(1/n) < \infty,$$

then the series (1) converges almost everywhere.

This theorem contains Theorems 1 and 2, (b) and (c). The method of the proof is that used to proved Theorem 1.

2. **Proof of Theorem 3.** We use a lemma due to A. Zygmund.

Lemma. *Let $p > 1$. If*

$$\left\|\sum_{\nu=m}^{n} \gamma_\nu e^{i\nu x}\right\|_p \leqq C,$$

$$|\lambda_\nu| < M, \qquad \sum_{\nu=m}^{n-1} |\lambda_\nu - \lambda_{\nu+1}| \leqq M,$$

then

$$\left\|\sum_{\nu=m}^{n} \gamma_\nu \lambda_\nu e^{i\nu x}\right\|_p \leqq A_p MC.$$

Let us now prove Theorem 3. It is sufficient to prove

$$\sum_{n=1}^{\infty}\int_0^{2\pi} |s_n(x)-f(x)|^{\lambda} dx < \infty.$$