171. *On a Multiple Exponential Sum*

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Let k be a finite field with $q=p^{\nu}$ elements and $k[x_1,\cdots,x_m]$ denote the ring of polynomials in m indeterminates $x_1,\cdots,x_m$ with coefficients in k. For $\alpha\in k$, we write as usual

$$e(\alpha)=e^{2\pi it(\alpha)/p},$$

where

$$t(\alpha)=\alpha+\alpha^p+\cdots+\alpha^{p^{\nu-1}}.$$

Given a polynomial $f=f(x_1,\cdots,x_m)\in k[x_1,\cdots,x_m]$ of degree n, not equivalent to a polynomial with indeterminates less than m in number, we construct the exponential sum

$$S_m(f)=\sum_{x_1,\cdots,x_m\in k}e(f(x_1,\cdots,x_m)),$$

where $x_1,\cdots,x_m$ run independently over all elements of k. It is assumed throughout that $1<n<p$.

Recently L. Carlitz and S. Uchiyama [1] have proved the inequality

$$(1)\qquad |S_1(f)|\leqq(n-1)q^{\frac{1}{2}},$$

which can be used, as we shall see, to obtain

$$(2)\qquad S_m(f)=O(q^{m-\frac{1}{2}}),$$

in general. Here and henceforth the constant implied by O depends only upon m and n. The inequality (2) may be compared with a result of S.-H. Min [3], who proved that

$$S_m(f)=O(q^{m\left(1-\frac{1}{n}\right)})$$

for a certain class of polynomials $f\in k[x_1,\cdots,x_m]$ of degree $n\geqq 2m$. Also, in the case of $m=2$, L.-K. Hua and S.-H. Min [2] proved that

$$S_2(f)=O(q^{2-\frac{2}{n}})$$

and that, if $n=3$, then

$$S_2(f)=O(q^{\frac{5}{4}}).$$

This last inequality is better than that in (2) with $m=2$, $n=3$.

Our proof of (2) is highly simple except for the use of the inequality (1). In fact, denoting by l the degree of the polynomial $f(x_1,\cdots,x_m)$ with respect to x_m, we write

$$f(x_1,\cdots,x_m)=\sum_{j=0}^{l}g_jx_m^{l-j},$$

where the $g_j=g_j(x_1,\cdots,x_{m-1})$ are polynomials independent of x_m. By the assumption, there exists, among the g_j $(0\leqq j\leqq l-1)$, one at least