164. *On the Cut Operation in Gentzen Calculi*

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By *Gentzen's LK system,* we shall mean a formal system with the following schemata:

Axiom schema

$$\mathfrak{A} \to \mathfrak{A}$$

Logical rules of inferences

	in succedent	in antecedent
$\supset$	$\dfrac{\mathfrak{A}, \Gamma \to \theta, \mathfrak{B}}{\Gamma \to \theta, \mathfrak{A} \supset \mathfrak{B}}$	$\dfrac{\Gamma \to \theta, \mathfrak{A} \quad \mathfrak{B}, \Gamma \to \theta}{\mathfrak{A} \supset \mathfrak{B}, \Gamma \to \theta}$
&	$\dfrac{\Gamma \to \theta, \mathfrak{A} \quad \Gamma \to \theta, \mathfrak{B}}{\Gamma \to \theta, \mathfrak{A} \& \mathfrak{B}}$	$\dfrac{\mathfrak{A}, \Gamma \to \theta}{\mathfrak{A} \& \mathfrak{B}, \Gamma \to \theta} \quad \dfrac{\mathfrak{B}, \Gamma \to \theta}{\mathfrak{A} \& \mathfrak{B}, \Gamma \to \theta}$
$\vee$	$\dfrac{\Gamma \to \theta, \mathfrak{A}}{\Gamma \to \theta, \mathfrak{A} \vee \mathfrak{B}} \quad \dfrac{\Gamma \to \theta, \mathfrak{B}}{\Gamma \to \theta, \mathfrak{A} \vee \mathfrak{B}}$	$\dfrac{\mathfrak{A}, \Gamma \to \theta \quad \mathfrak{B}, \Gamma \to \theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \to \theta}$
$\neg$	$\dfrac{\mathfrak{A}, \Gamma \to \theta}{\Gamma \to \theta, \neg \mathfrak{A}}$	$\dfrac{\Gamma \to \theta, \mathfrak{A}}{\neg \mathfrak{A}, \Gamma \to \theta}$
$\forall$	$\dfrac{\Gamma \to \theta, \mathfrak{A}(a)}{\Gamma \to \theta, \forall x \mathfrak{A}(x)}$	$\dfrac{\mathfrak{A}(t), \Gamma \to \theta}{\forall x \mathfrak{A}(x), \Gamma \to \theta}$
$\exists$	$\dfrac{\Gamma \to \theta, \mathfrak{A}(t)}{\Gamma \to \theta, \exists x \mathfrak{A}(x)}$	$\dfrac{\mathfrak{A}(a), \Gamma \to \theta}{\exists x \mathfrak{A}(x), \Gamma \to \theta}$

Structural rules of inferences

	in succedent	in antecedent
Thinning	$\dfrac{\Gamma \to \theta}{\Gamma \to \theta, \mathfrak{A}}$	$\dfrac{\Gamma \to \theta}{\mathfrak{A}, \Gamma \to \theta}$
Contraction	$\dfrac{\Gamma \to \theta, \mathfrak{A}, \mathfrak{A}}{\Gamma \to \theta, \mathfrak{A}}$	$\dfrac{\mathfrak{A}, \mathfrak{A}, \Gamma \to \theta}{\mathfrak{A}, \Gamma \to \theta}$
Interchange	$\dfrac{\Gamma \to \Lambda, \mathfrak{A}, \mathfrak{B}, \theta}{\Gamma \to \Lambda, \mathfrak{B}, \mathfrak{A}, \theta}$	$\dfrac{\Gamma, \mathfrak{A}, \mathfrak{B}, \theta \to \Lambda}{\Gamma, \mathfrak{B}, \mathfrak{A}, \theta \to \Lambda}$

Cut
$$\frac{\Delta \to \Lambda, \mathfrak{A} \quad \mathfrak{A}, \Gamma \to \theta}{\Delta, \Gamma \to \Lambda, \theta}.$$

In the above schemata, $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \cdots$ are arbitrary formulae, $a, b, c, \cdots$ are free variables and $s, t, \cdots$ are terms. $\Gamma, \Delta, \theta, \cdots$ are arbitrary finite sequences of zero or more formulae. *For the detail on the schemata,* see G. Gentzen [1], S. C. Kleene [2, 3].

In his papers, G. Gentzen [1] proved a principal theorem: any provable proposition in *LK*-system is provable without cut. And he observed that *the cut is equivalent to the mix rule*: let $\mathfrak{A}$ be a formula,