3. *Complex Numbers with Vanishing Power Sums*

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1. By $\mathfrak{Z}_{m,n}$ we denote the set of systems of n complex numbers $(z_1, z_2, \cdots, z_n)$ with the property

$$s_\nu \equiv \sum_{j=1}^{n} z_j^\nu = 0 \qquad (\nu = m+1, m+2, \cdots, m+n-1)$$

for a prescribed non-negative integer m.

In a course of their study of the theory of Diophantine approximations Vera T. Sós and P. Turán*) were led to the problem of determining all the systems in $\mathfrak{Z}_{m,n}$, and proved that:

1° the systems in $\mathfrak{Z}_{0,n}$ are given by the zeros of an equation

$$z^n + a = 0 \qquad (a \text{ arbitrary complex});$$

2° the systems in $\mathfrak{Z}_{1,n}$ are given by the zeros of an equation

$$z^n + \frac{a}{1!} z^{n-1} + \cdots + \frac{a^n}{n!} = 0 \quad (a \text{ arbitrary complex}); \text{ and}$$

3° the systems in $\mathfrak{Z}_{2,n}$ are formed by the zeros of an equation

$$z^n + \frac{H_1(\lambda)}{1!} a z^{n-1} + \cdots + \frac{H_n(\lambda)}{n!} a^n = 0,$$

where $H_\nu(t)$ stands for the νth Hermite polynomial defined by

$$H_\nu(t) = (-1)^\nu e^{t^2} \frac{d^\nu}{dt^\nu} e^{-t^2},$$

λ denotes any zero of the equation $H_{n+1}(t) = 0$ and a is an arbitrary complex number.

In the present note we wish to give a characterization of the systems in $\mathfrak{Z}_{m,n}$ for general integer values of $m > 0$.

2. We define polynomials $C_\nu = C_\nu(t_1, \cdots, t_m)$ $(\nu = 0, 1, 2, \cdots)$ by

$$(1) \qquad \exp\left(-\sum_{\mu=1}^{m} \frac{1}{\mu} t_\mu x^\mu \right) = \sum_{\nu=0}^{\infty} \frac{C_\nu}{\nu!} x^\nu,$$

that is, by

$$C_\nu = \nu! \sum_{\substack{\mu_i \geq 0 \\ \mu_1 + 2\mu_2 + \cdots + m\mu_m = \nu}} \frac{\left(-\frac{t_1}{1}\right)^{\mu_1} \left(-\frac{t_2}{2}\right)^{\mu_2} \cdots \left(-\frac{t_m}{m}\right)^{\mu_m}}{\mu_1! \, \mu_2! \cdots \mu_m!}$$

It is well known that the Hermite polynomials $H_\nu(t)$ $(\nu = 0, 1, 2, \cdots)$ are generated by

$$e^{2tx - x^2} = \sum_{\nu=0}^{\infty} \frac{H_\nu(t)}{\nu!} x^\nu.$$

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*) Vera T. Sós and P. Turán: On some new theorems in the theory of Diophantine approximations, Acta Math. Acad. Sci. Hungar., **6**, 241–255 (1955).