25. On Pseudo-compact and Countably Compact Spaces

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In his kind letter of January 13, 1957 to S. Kasahara, one of the present writers, Prof. S. Mardešić of the University of Zagreb, Yugoslavia, communicated an interesting characterisation of pseudo-compact without proof by S. Mrówka. The result stated which is due to him is the following

*Theorem. A completely regular space is pseudo-compact if and only if every locally finite open covering has a finite subcovering.**)

The concept of pseudo-compact space was introduced by E. Hewitt [2]. A completely regular space is said to be *pseudo-compact*, if every real continuous function on it is bounded.

In this Note, we shall first give a simple proof of Theorem. To prove it, we shall prove the following

Theorem 1. The following properties of a completely regular space S are equivalent:

(1) *S is pseudo-compact.*

(2) *Every locally finite open covering has a finite subcovering.*

(3) *Every star finite open covering has a finite subcovering.*

Proof. To prove the implication (1)$\rightarrow$(2), let $\sigma=\{O_\alpha\}$ be a locally finite open covering of S. Suppose that σ has no finite subcovering, then we can find a denumerable subfamily $\{O_n\}$ of σ which every finite family of it does not cover S. With each O_n, we associate a certain point $a_n \in O_n$. Since S is completely regular, for every n, we can find a non-negative continuous function $f_n(x)$ such that $f_n(a_n)=n$ and $f_n(x)=0$ for $x\in S-O_n$. Since σ is locally finite $f(x)=\sum_{n=1}^{\infty} f_n(x)$ is well-defined and continuous on S. On the other hand, $f(a_n)\geq n$, and hence $f(x)$ is unbounded continuous, which is a contradiction to the hypothesis. Therefore we have (1)$\rightarrow$(2).

The implication (2)$\rightarrow$(3) is trivial, since every star finite open covering is locally finite.

To prove (3)$\rightarrow$(1), we shall show that any non-negative continuous function $f(x)$ is bounded. It is obvious that it leads the pseudo-compactness of S. By the continuity of $f(x)$, the sets $O_1=\{x\mid f(x)<2\}$, $O_n=\{x\mid n-1<f(x)<n+1\}$ $(n=2,3,\cdots)$ are open. The family $\{O_n\}$ is

*) For various terminologies, see J. L. Kelley: General Topology, New York (1955).