17. On Hardy and Littlewood's Theorem

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1. Let $f(x)$ be an L-integrable function with period 2π, and its Fourier series be

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx). \tag{1}$$

A. Zygmund [1] has shown the following

Theorem Z. *If $f(x)$ belongs to $Lip\,\alpha$ where $0<\alpha\leqq 1$, then the series (1) is uniformly summable $(C, -\alpha+\delta)$ to $f(x)$ for every $\delta>0$.*

Later, Hardy and Littlewood [2] showed the following

Theorem H., L. *If $f(x)$ belongs to $Lip\,(\alpha, p)$ where $0<\alpha\leqq 1$ and $\alpha p>1$, i.e.*

$$\left(\int_0^{2\pi}|f(x+h)-f(x)|^p dx\right)^{1/p}=O(|h|^{\alpha})$$

as $h\to 0$, then the series (1) is uniformly summable $(C, -\alpha+\delta)$ to $f(x)$ for every $\delta>0$.

In this paper we shall improve the above theorem as follows:

Theorem. *If $f(x)$ is continuous in $(0, 2\pi)$, and belongs to $Lip\,(\alpha, 1/\alpha)$ where $0<\alpha\leqq 1$, i.e.*

$$\int_0^{2\pi}|f(x+h)-f(x)|^{1/\alpha}dx=O(h)$$

as $h\to 0$, then the series (1) is uniformly summable $(C, -\alpha+\delta)$ to $f(x)$ for every $\delta>0$.

2. The proof*) of our theorem is as follows. Let

$$\varphi(t)=\varphi_x(t)=f(x+t)+f(x-t)-2f(x),$$

then we have

$$\varphi(t)\to 0 \text{ as } t\to 0 \text{ uniformly in } 0\leqq x\leqq 2\pi, \tag{2}$$

since f is continuous.

We denote the n-th (C,γ) mean of the series (1) by $\sigma_n^{\gamma}(x)$, then

$$\sigma_n^{-\alpha}(x)-f(x)=\frac{1}{\pi}\int_0^{\pi}\varphi(t)K_n^{-\alpha}(t)dt$$

$$=\frac{1}{\pi}\int_0^{K/n}+\frac{1}{\pi}\int_{K/n}^{\pi}=I_1+I_2$$

say, where $K_n^{\gamma}(t)$ is the n-th (C,γ) Féjer kernel and

$$|K_n^{-\alpha}(t)|\leqq\frac{n}{1-\alpha}+\frac{1}{2}\quad\text{for } 0\leqq t\leqq\pi, \tag{3}$$

*) The method of this proof has been suggested to me by Prof. G. Sunouchi.