15. Fourier Series. XII. Bernstein Polynomials

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(Comm. by Z. SUETUNA, M.J.A., Feb. 12, 1957)

1. If $f(t)$ is integrable in the closed interval $[0, 1]$, then the generalized Bernstein polynomials of $f(t)$ are defined as

$$(1) \qquad P_n(x)=P_n(x,f)=\sum_{\nu=0}^{n}(n+1)p_{n,\nu}(x)\int_{\nu/(n+1)}^{(\nu+1)/(n+1)}f(t)dt \qquad (n=0,1,2,\cdots),$$

where

$$(2) \qquad p_{n,\nu}(x)=p_{n,\nu}=\binom{n}{\nu}x^{\nu}(1-x)^{n-\nu}.$$

It is known that $P_n(x,f)$ tends to $f(x)$ almost everywhere as $n\to\infty$ and carries many properties of the Fejér mean of the Fourier series of $f(t)$ [1]. From this point of view P. L. Butzer [2] considered the polynomials, corresponding to the partial sums of the Fourier series of $f(t)$ such that

$$(3) \qquad Q_n(x)=Q_n(x,f)=(n+1)P_n(x,f)-nP_{n-1}(x,f) \qquad (n=0,1,2,\cdots),$$

and established some fundamental theorems concerning them.

Among others he proved the following

Theorem 1. *If $f(t)$ is bounded in the interval $(0, 1)$ and its second derivative exists at $t=x$, then $Q_n(x,f)$ tends to $f(x)$ as $n\to\infty$.*

Further he raised the question:

Does there exist an integrable function $f(t)$ such that the $Q_n(x,f)$ diverges almost everywhere in the interval $(0, 1)$?

In the present note we wish to prove the following theorems:

Theorem 2. *If the derived Fourier series of $f(t)$ converges absolutely, then $Q_n(x,f)$ converges to $f(x)$ everywhere.*

Theorem 3. *There is a continuous function $f(t)$ with absolutely convergent Fourier series such that $Q_n(x,f)$ diverges almost everywhere.*

Clearly Theorem 3 is a stronger solution of the problem of Butzer's. We note that, as will be found incidentally in § 3, our Theorem 2 can not hold in general unless the derived Fourier series of $f(t)$ is absolutely convergent.

2. **Proof of Theorem 2.** Without loss of generality we may suppose that

$$f(t)\sim\sum_{\lambda=1}^{\infty}a_{\lambda}e^{2\pi i\lambda t}$$

Then

$$(4) \qquad Q_n(x,f)-f(x)=\sum a_{\lambda}[Q_n(x,e^{2\pi i\lambda t})-e^{2\pi i\lambda x}]$$