50. Boundedness of Semicontinuous Finite Real Functions

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A number of interesting characterizations of pseudo-compact spaces[1] has been given by various authors. In the present note, we concern with the spaces on which all semicontinuous finite valued real functions are bounded. All topological spaces to be considered in what follows will be assumed to satisfy the axiom T_1 of Fréchet.

THEOREM 1. *The following properties of a topological space E are equivalent:*

(1) *Every upper semicontinuous finite real function on E is bounded above.*[2]

(2) *Every lower semicontinuous finite real function on E is bounded below.*

(3) *The space E is countably compact.*[3]

Proof. It is clear that (1) and (2) are equivalent. To prove the implication (1)$\to$(3), suppose that E is not countably compact. Then there exists a sequence $\{x_n\}$ $(n=1,2,\cdots)$ of points of E which has no cluster point. A set which consists of a single point being closed, the function f_n defined by

$$f_n(x)=\begin{cases} n & \text{for } x=x_n \\ 0 & \text{for } x\neq x_n \end{cases}$$

is upper semicontinuous for any n. As can readily be seen, the function $f(x)=\sum_{n=1}^{\infty} f_n(x)$ is of finite value. Since the subsequence $\{x_n\}_{n\geq m}$ is a closed set for any positive integer m, f is upper semicontinuous, but it is not bounded above. Conversely, let us suppose that the space E is countably compact, and let an upper semicontinuous finite real function f on E be given. Then, for any positive integer n, the set $O_n=\{x\in E; f(x)<n\}$ being open, the sets O_n form a countable open covering of E when n runs over the positive integers. Now, since E is countably compact, for a suitable positive integer m, we have $E=O_m$, that is to say $f(x)<m$ for all $x\in E$. This completes the proof of the theorem.

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1) A completely regular space is said to be *pseudo-compact* if every continuous function on it is bounded.

2) We say that a real valued function f defined on a set E is *bounded above* (*bounded below*) if there is a constant k such that $f(x)\leq k$ $(f(x)\geq k)$ for all $x\in E$.

3) A topological space in which every countable open covering has a finite subcovering is called *countably compact*.