47. On Open Mappings

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1. Let X and Y be topological spaces and let f be a mapping of X onto Y. f is said to be open (closed) if the image of every open (closed) subset of X is open (closed) in Y. It is easy to see that if f is an open continuous mapping, then the local compactness is invariant under f. As a generalization of the notion of the local compactness, L. Zippin [4] has introduced the notion of the semicompactness.

In the present note, we shall consider conditions under which the semicompactness is invariant under an open continuous mapping. The conditions we obtain are sufficient; however, we show by an example that if we drop one of the conditions the semicompactness is not always invariant.

2. We begin with giving the definition of the semicompactness introduced by L. Zippin. A topological space X is called semicompact at a point x if every neighborhood U of x contains an open neighborhood V of x such that the boundary $\mathfrak{B}V$ is compact.[1] X is called semicompact if X has this property at every point.

Theorem 1. *Let f be an open continuous mapping of a topological space X onto a topological space Y. If f is closed, then the semicompactness is invariant under the mapping f.*

Proof. Let y be any point of Y and let $U(y)$ be any open neighborhood of y. Then $f^{-1}\{U(y)\}$ is an open set containing the inverse image $f^{-1}(y)$ since f is continuous. Let x be any point of $f^{-1}(y)$, then there exists an open neighborhood $O(x)$ such that $O(x)\subset f^{-1}\{U(y)\}$ and $\mathfrak{B}O(x)$ is compact since X is semicompact. Since f is an open and closed continuous mapping and $O(x)$ is an open set, $f\{O(x)\}$ is an open neighborhood of y such that $f\{O(x)\}\subset U(y)$ and $\mathfrak{B}f\{O(x)\}\subset f\{\mathfrak{B}O(x)\}$ (see G. T. Whyburn [2], p. 147). Since f is closed and continuous, $f\{\mathfrak{B}O(x)\}$ is a closed compact set in Y. Hence $\mathfrak{B}f\{O(x)\}$ is compact. Therefore Y is a semicompact space. Thus Theorem 1 is proved.

Theorem 2. *Let f be an open continuous mapping of a Hausdorff space X onto a weakly separable[2] Hausdorff space Y such that the inverse image $f^{-1}(y)$ is connected for every point y of Y. If the*

1) According to L. Zippin, we allow that $\mathfrak{B}V$ may be vacuous. A semicompact space is also said to be locally peripherally compact (cf. [1]).

2) A topological space satisfying the first axiom of countability will be called weakly separable.