45. *Some Examples of (F) and (DF) Spaces*

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In this paper we answer to the questions of A. Grothendieck [1] (question 1, 4, and 5 partly) giving some negative examples.

We say that an (F) space E has the *stability of the boundedness*, if for every bounded set B of E and every dense subspace E_0 of E, there exists a bounded set A of E_0 such that the closure of A includes B.

We show in §1 that there exists an (F) space which has not the stability of the boundedness. In §2, we give an example of reflexive (F) space in whose dual there exists a bounded set on which the strong topology is not metrizable. In §3, an example of bornologic (DF) space whose bidual is not bornologic is given.

§1. Let E be an (F) space, F a Banach space, and u a linear operator defined on a dense subspace E_0 of E into F, such that (1) $u(E_0)$ is dense in F, (2) $u^{-1}(0)$ is dense in E_0, and (3) for any bounded set B of E_0, the closure of $u(B)$ is not a neighbourhood of 0 in F. Then the graph of u, $G=\{(x, u(x)); x\in E_0\}$, in $E\times F$ is a dense subspace of $E\times F$, and the closure of any bounded set A of G does not include the unit sphere U of F considered as a subspace of $E\times F$. In fact, we have $A\subset B\times u(B)$ for the image B of A by the projection of $E\times F$ to E, while the closure of $B\times u(B)$ does not include U by the condition (3).

Thus $E\times F$ has not the stability of the boundedness, so we have only to give an example of such E, F and u.

Let S be a non-normable (F) space and B_λ $(\lambda\in\Lambda)$ a basis of bounded sets of S. Then we can find, for every $\lambda\in\Lambda$, a non-continuous linear functional $\varphi_\lambda\neq 0$ on S such that $\varphi_\lambda(x)=0$ for every $x\in B_\lambda$. Let E be $l^1(\Lambda, S)$, i.e. the usually defined (F) space of all functions $f(\lambda)$ of Λ into S such that $\sum_{\lambda\in\Lambda} f(\lambda)$ is absolutely summable in S. For $f\in E$, we define $u(f)$ as a function on Λ of which the value at λ is $\varphi_\lambda(f(\lambda))$. Now put $F=l^1(\Lambda)$, and $E_0=\{f; u(f)\in F\}$, then E_0 is dense in E and u, restricted on E_0, satisfies the conditions (1), (2) and (3).

In fact, $u(E_0)$ contains every function whose values are 0 except for a finite number of λ, and hence, is dense in F. The condition (2) is also satisfied, since $\varphi_\lambda^{-1}(0)$ is dense in S. Let A be an arbitrary bounded set of E_0, then there exists $\lambda_0\in\Lambda$ such that $f\in A$ implies $f(\lambda)\in B_{\lambda_0}$ for every $\lambda\in\Lambda$. Then the value of $u(f)$ at λ_0 is 0 for every $f\in A$, and hence the closure of $u(A)$ is not a neighbourhood of 0 in