67. *Perturbation of Continuous Spectra by Trace Class Operators*

By Tosio KATO
Department of Physics, University of Tokyo
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1. Introduction. In a previous paper[1] the writer has shown, among others, that the absolutely continuous part of the spectrum of a self-adjoint operator is stable under a self-adjoint perturbation of finite rank. The purpose of the present note is to extend this result to a wider class of perturbations.

Let $\mathfrak{H}$ be a Hilbert space, $\mathbf{B}$ the algebra of all bounded linear operators on $\mathfrak{H}$ to $\mathfrak{H}$, $\mathbf{S}\subset\mathbf{B}$ the Schmidt class and $\mathbf{T}\subset\mathbf{S}$ the trace class.[2] We denote by $\|\ \|$, $\|\ \|_2$, $\|\ \|_1$ the ordinary norm, the Schmidt norm and the trace norm respectively. The subset of $\mathbf{T}$ consisting of all self-adjoint operators will be denoted by $\mathbf{T}_s$.

THEOREM 1. *Let H_0 be a (not necessarily bounded) self-adjoint operator and let $V\in\mathbf{T}_s$. Then $H_1=H_0+V$ is also self-adjoint. Let $\mathfrak{M}_0$ and $\mathfrak{M}_1$ be the absolutely continuous*[3] *parts of $\mathfrak{H}$ with respect to H_0 and H_1, and let P_0 and P_1 be the projections on $\mathfrak{M}_0$ and $\mathfrak{M}_1$, respectively. Then the strong limits*

$$s-\lim_{t\to\pm\infty}\exp(itH_1)\exp(-itH_0)P_0=U_\pm \tag{1.1}$$

exist and are partially isometric operators with initial set $\mathfrak{M}_0$ and final set $\mathfrak{M}_1$. Their adjoints satisfy

$$s-\lim_{t\to\pm\infty}\exp(itH_0)\exp(-itH_1)P_1=U_\pm^*. \tag{1.2}$$

The parts of H_0 and H_1 in $\mathfrak{M}_0$ and $\mathfrak{M}_1$ respectively are unitarily equivalent to each other, and the transformation of H_0P_0 into H_1P_1 is effected by either of $U_\pm$:[4]

$$H_1P_1=U_\pm H_0P_0U_\pm^*,\qquad H_0P_0=U_\pm^*H_1P_1U_\pm. \tag{1.3}$$

The second theorem concerns itself with the properties of the mappings which assign to each pair H_0, H_1 the operators $U_\pm$ by (1.1). Let $\mathbf{H}$ be any one of the equivalence classes of self-adjoint operators

1) T. Kato: On finite-dimensional perturbations of self-adjoint operators, J. Math. Soc. Japan, **9**, 239–249 (1957). This paper is quoted as (F).

2) For the Schmidt and trace classes, see R. Schatten: A theory of cross spaces, Ann. Math. Studies, Princeton (1950).

3) For the terms "absolutely continuous" as applied to operators and vectors, see (F).

4) Theorem 1 contains as a special case a theorem by M. Rosenblum in his paper "Perturbation of the continuous spectrum and unitary equivalence", to be published in Pacific J. Math. The writer is indebted to Professor Rosenblum for having a chance of seeing his paper before formal publication.