83. On Kählerian Manifolds with Positive Holomorphic Sectional Curvature

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S. B. Myers [2] proved that any complete Riemannian manifold with positive sectional curvature is compact. And J. L. Synge [3] proved that if an orientable even dimensional Riemannian manifold is complete and has positive sectional curvature, then it is simply connected. The purpose of this note is to generalize these results to Kählerian manifolds*) with positive holomorphic sectional curvature.

A Kählerian manifold M is said to have positive holomorphic sectional curvature greater than or equal to a positive number e if the holomorphic sectional curvature of M is greater than or equal to e at all points and for all directions. In this note we consider a real representation of Kählerian manifold, that is, the Riemannian manifold with the parallel complex structure.

Theorem 1. A complete n-dimensional Kählerian manifold M whose holomorphic sectional curvature is greater than or equal to $e>0$ is compact and has the diameter less than or equal to $\pi/\sqrt{e}$.

Proof. Let I be the tensor of type (1, 1) which defines the complex structure of M. Let A and B be any two points of M and g be a minimizing geodesic arc which binds A and B. Let X_A be the unit tangent vector to g at A. We can obtain the parallel vector field X on g by displacing X_A parallely along g.

Since I is a parallel tensor field on M, IX is parallel and moreover it is perpendicular to X on g. Let p be an arbitrary point on g and π_p be the tangent subspace of M at p which is spanned by X_p and $(IX)_p$.

Now we can construct a 2-dimensional surface S in a neighborhood of g which has π_p as tangent space at p, $p\in g$.

We induce Riemann metric of M to S. Then, according to Synge's theorem (cf. Chern [1, p. 137]), the Gaussian curvatures of S are equal at all points of g to the sectional curvatures in the corresponding tangent plane π_p of S. So these Gaussian curvatures are greater than or equal to e.

By Sturm's comparison theorem (cf. Myers [2]), the length of g can not be greater than $\pi/\sqrt{e}$. So the diameter of M is less than or equal to e and hence M is compact.

Q.E.D.

*) They hold good for pseudo-Kählerian manifolds, too.