79. A Characterisation of Pseudo-compact Spaces

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Recently, T. Isiwata [5] has given some characterisations of a countably compact normal space by the concept of quasi-uniform continuity. In this Note, we shall give a characterisation of a pseudo-compact completely regular space by locally uniformly convergence. Some types of the convergences of a sequence of functions have been known (see H. Hahn [1, pp. 211-231]).

Let S be a completely regular T_1-space, and suppose that $f_n(x)$ $(n=1, 2, \cdots)$ and $f(x)$ are real valued continuous functions on S. The sequence $f_n(x)$ is said to *converge uniformly at a point* x_0 of S, if, for every $\varepsilon>0$, there are an index N and a neighbourhood U of x_0 such that $|f_n(x)-f(x)|<\varepsilon$ for all $n\geq N$ and $x \in U$. Then we shall prove the following

Theorem 1. Let S be a completely regular T_1-space. Then S is pseudo-compact if and only if any sequence $\{f_n(x)\}$ of continuous functions which converges uniformly to a continuous function $f(x)$ at every point of S converges uniformly in S to $f(x)$.

For the concept of pseudo-compactness, see E. Hewitt [2, p. 67].

Proof. Suppose that S is not pseudo-compact, then there is an unbounded continuous function $f(x)$. For each positive integer n, we shall define $f_n(x)$ as

$$f_n(x) = \mathrm{Min}\,(f(x), n),$$

then it is obvious that $f_n(x)$ is continuous.

For a point x_0 of S, by the continuity of $f(x)$, we can find a neighbourhood U of x_0 and a positive integer N such that $f(x)<N$ for all x of U.

Therefore, by the definition of $f_n(x)$, we have $f_n(x)=f(x)$ for x of U and all $n\geq N$. Hence $f_n(x)$ converges uniformly at the point x_0 to $f(x)$. On the other hand, for every $\varepsilon>0$, we can find a point x_0 such that $|f(x_0)-f_n(x_0)|>\varepsilon$, since $f(x)$ is unbounded. Hence $f_n(x)$ does not converge uniformly to $f(x)$.

To prove the converse, we shall use a theorem of S. Mardešić and P. Papić [6]: *A completely regular T_1-space is pseudo-compact, if and only if every countable open covering has AU-covering* (for the definition of AU-covering, see K. Iséki [3]). Let S be a pseudo-compact, completely regular T_1-space, and suppose that $\{f_n(x)\}$ coverges uniformly at every point to $f(x)$. For a given positive $\varepsilon>0$, and each