# 77. Note on Orlicz-Birnbaum-Amemiya's Theorem

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W. Orlicz and Z. Birnbaum proved in [2] that an Orlicz space $L_\Phi(G)$ is finite if and only if the function $\Phi$ satisfies the following condition for some $\gamma>0$:

$$\Phi(2t)\leqq\gamma\Phi(t) \qquad \text{for every} \quad t\geqq t_0.$$

(In case of $mes(G)=+\infty$, $\Phi(2t)\leqq\gamma\Phi(t)$ for all $t\geqq 0$.)

This fact was generalized for arbitrary monotone complete modulars[1] on non-discrete spaces by I. Amemiya in [1] recently. In this note we shall show a new simple proof to this Amemiya's theorem.

As for an Orlicz-sequence space $l_\Phi$, W. Orlicz and Z. Birnbaum also proved in [2] that $l_\Phi$ is finite if and only if the function $\Phi$ satisfies the following condition for some $\gamma>0$:

$$\Phi(2t)\leqq\gamma\Phi(t) \qquad \text{for every} \quad 0\leqq t\leqq t_0.$$

We shall generalize this fact on arbitrary modulars on discrete spaces.

§1. Let $R$ be a universally continuous semi-ordered space and $m$ be a modular on $R$. A modular is said to be "*finite*", if $m(x)<+\infty$ for every $x\in R$. And a modular on $R$ is said to be "*semi-upper bounded*", if for every $\varepsilon>0$ there exists $\gamma_\varepsilon$ ($\gamma_\varepsilon>0$) such that $m(x)\geqq\varepsilon$ implies $m(2x)\leqq\gamma_\varepsilon m(x)$. Now we shall prove

**Theorem 1** (I. Amemiya). *Suppose that $R$ has no atomic element, then every monotone complete finite modular on $R$ is semi-upper bounded.*

**Proof.** We shall prove first that there exists $\gamma_1$ such that $m(x)\geqq 1$ implies $m(2x)\leqq\gamma_1 m(x)$. If such $\gamma_1$ can not be found, then we can find a sequence of elements $0\leqq x_\nu\in R$ ($\nu=1,2,\cdots$) such that

(1) $\qquad m(2x_\nu)>\nu 2^{\nu+1}m(x_\nu),\ N_\nu\leqq m(x_\nu)\leqq N_\nu+1 \ (\nu=1,2,\cdots),$

where $N_\nu$ ($\nu\geqq 1$) is a natural number.

(1) implies immediately

(2) $\qquad m(2x_\nu)>\nu 2^\nu(N_\nu+1) \qquad (\nu=1,2,\cdots).$

Since $R$ has no atomic element, $x_\nu$ can be decomposed orthogonally as $x_\nu=\sum_{\mu=1}^{(N_\nu+1)2^\nu}x_{\nu,\mu}$, $m(x_{\nu,\mu})=m(x_{\nu,\rho})$ ($\mu,\rho=1,2,\cdots,(N_\nu+1)2^\nu$) for every $\nu\geqq 1$. As $m(x_\nu)<N_\nu+1$, we have $m(x_{\nu,\mu})\leqq\frac{1}{2^\nu}$ for every $1\leqq\mu\leqq 2^\nu(N_\nu+1)$.

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1) For the definition of the modular see H. Nakano [3]. A modular $m$ is said to be monotone complete, if $0\leqq a_\lambda\uparrow_{\lambda\in\Lambda}$, $\sup_{\lambda\in\Lambda}m(a_\lambda)<+\infty$ implies the existence of $\bigcup_{\lambda\in\Lambda}a_\lambda$.