76. On the Norms by Uniformly Finite Modulars

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Let R be a modulared semi-ordered linear space and m be a modular[1] on R. On R we can define two norms as follows:

$$\|x\| = \inf_{\xi>0} \frac{1+m(\xi x)}{\xi}, \quad |||x||| = \inf_{m(\xi x)\leqq 1} \frac{1}{|\xi|} \qquad (x \in R).$$

$\|x\|$ is said to be the first norm by m and $|||x|||$ is said to be the second norm or the modular norm by m. Since we have $|||x|||\leqq\|x\| \leqq 2|||x|||$ for every $x\in R$ (cf. [4]), they are equivalent to each other.

It is well known that if a modular m is finite, i.e. $m(x)<+\infty$ for all $x\in R$, then the modular norm is continuous, and that the converse of this is true when R has no atomic element.

In [1] I. Amemiya showed that if a modular m is monotone complete[2] and the modular norm is continuous, then the norm satisfies the following condition: *for every* $1>\varepsilon>0$ *there exists an integer* n *such that the norm of the sum of* n *mutually orthogonal elements having their norm more than* ε *is always* $\geqq 1$. In this paper we call the norm satisfying the above condition to be *finitely monotone.*

We shall investigate the properties of finitely monotone norm and show the form of the conjugate norm in §1. In §2 we examine the relations between a modular and the modular norm in case it is finitely monotone. In fact, we shall prove that if a modular m is uniformly finite, then the modular norm is finitely monotone. The converse of this is valid, if we suppose that R has no atomic element.

If a modular is defined on a universally continuous semi-ordered linear space, then as showed above, we can define the norms whose convergences are equivalent to the modular convergence.[3] Thus it will be conjectured that if a norm is defined on a universally continuous semi-ordered linear space, then there may be defined a modular whose convergence is equivalent to the norm convergence. In §3 we shall establish a normed semi-ordered linear space which is a sort of Köthe space on [0, 1], and it answers negatively to this conjecture. Finitely

1) For the definition of the modular see H. Nakano [4]. The notations and terminologies used here are due to the book [4].

2) m is said to be monotone complete if $0\leqq a_\lambda \uparrow_{\lambda\in\Lambda}$, $\sup_{\lambda\in\Lambda} m(a_\lambda)<+\infty$ implies the existence of $\bigcup_{\lambda\in\Lambda} a_\lambda$.

3) A sequence of elements $x_i\in R$ $(i=1,2,\cdots)$ is said to be modular convergent to x_0, if $\lim_{i\to\infty} m(\xi(x_i-x_0))=0$ for every $\xi>0$.